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On (4,2)-digraphs Containing a Cycle of Length 2

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Abstract A *diregular* digraph is a digraph with the in-degree and out-degree of all vertices is constant. The *Moore* bound for a diregular digraph of degree d and diameter k is $M_{d,k} = 1+d+d^2+\cdots+d^k$. It is well known that diregular digraphs of order $M_{d,k}$, degree $d>1$ and diameter $k>1$ do not exist. A (d,k)-digraph is a diregular digraph of degree $d>1$, diameter $k>1$, and number of vertices one less than the *Moore* bound. For degrees $d=2$ and 3, it has been shown that for diameter $k \geq 3$ there are no such (d, k)-digraphs. However for diameter 2, it is known that $(d,2)$-digraphs do exist for *any* degree d. The line digraph of K_{d+1} is one example of such $(d,2)$-digraphs. Furthermore, the recent study showed that there are three non-isomorphic (2,2)-digraphs and exactly one non-isomorphic (3,2)-digraph. In this paper, we shall study (4,2)-digraphs. We show that if (4,2)-digraph G contains a cycle of length 2 then G must be the line digraph of a complete digraph K_5.

1. Introduction

A *digraph G* is a system consisting of a finite nonempty set $V(G)$ of objects called *vertices* and a set $E(G)$ of ordered pairs of distinct vertices called *arcs*. The *order* of G is the cardinality of $V(G)$. A *subdigraph H* of G is a digraph having all vertices and arcs in G. If (u,v) is an arc in a digraph G, then u is said to be *adjacent to* v and v is said to be *adjacent from u*. An *in-neighbor* of a vertex v in a digraph G is a vertex u such that $(u,v)\in G$. An *out-neighbor* of a vertex v in a digraph G is a vertex w such that $(v,w)\in G$. The set of all out-neighbors of a vertex v is denoted by $N^+(v)$ and its cardinality is called the *out-degree* of v, $d^+(v) = |N^+(v)|$. Similarly, the set of all in-neighbors of a vertex v is denoted by $N^-(v)$ and its cardinality is called the *in-degree* of v, $d^-(v) = \left|N^-(v)\right|$. A digraph G is *diregular* of degree d if for any vertex v in G, $d^+(v) = d^-(v) = d$.

A *walk* of length h from a vertex u to vertex v in G is a sequence of vertices $(u=u_0, u_1, \cdots, u_h = v)$ such that $(u_{i-1}, u_i)\in G$ for each i. A vertex u forms the *trivial* walk of length 0. A *closed walk* has $u_0 = u_h$. A *path* is a walk in which all points are distinct. A *cycle* C_h of length $h > 0$ is a closed walk with *h* distinct vertices (except u_0 and u_h). If there is a path from *u* to *v* in *G* then we say that *v* is *reachable* from *u*.

The *distance* from vertex *u* to vertex *v* in a digraph *G*, denoted by $\delta(u,v)$, is defined as the length of a shortest path from *u* to *v*. In general, $\delta(u,v)$ is not necessarily equal to $\delta(v,u)$. The *diameter k* of a digraph *G* is the maximum distance between any two vertices in *G*.

Let *G* be a diregular digraph of degree *d* and diameter *k* with *n* vertices. Let one vertex be distinguished in *G*. Let $n_i, \forall i = 0, 1, \cdots, k$, be the number of vertices at distance *i* from the distinguished vertex. Then,

$$n_i \leq d^i, \text{ for } i = 1, \cdots, k. \tag{1}$$

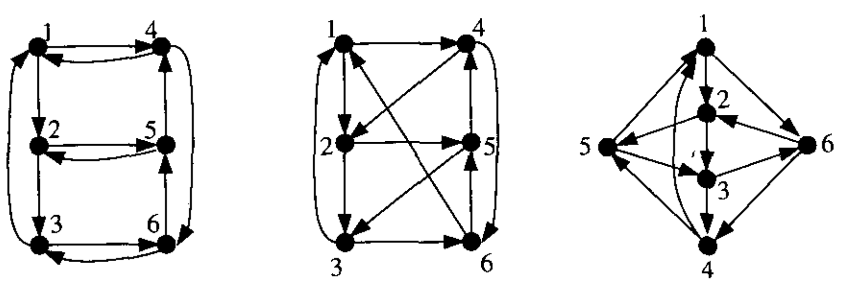
Hence,

$$n = \sum_{i=0}^{k} n_i \leq 1 + d + d^2 + \cdots + d^k. \tag{2}$$

The number of $1 + d + d^2 + \cdots + d^k$ is the upper bound for the number of vertices in digraph *G*. This upper bound is called *Moore bound* and denoted by $M_{d,k}$. If the equality sign in (2) holds then the digraph *G* is called *Moore digraph*.

It has been known that the Moore digraphs do not exist for $d > 1$ and $k > 1$, except for trivial cases (for $d = 1$ or $k = 1$), [10] and [5]. The trivial cases are fulfilled by the cycle digraph C_{k+1} for $d = 1$, and the complete digraph K_{d+1} for $k = 1$. This motivates the study of the existence problem of diregular digraphs of degree *d*, diameter *k* with order $M_{d,k} - 1$. Such digraphs are called *Almost Moore digraphs* and denoted by *(d,k)-digraphs*.

Several results have been obtained on the existence of (*d*,*k*)-digraphs. For instance, in [6] it is shown that the (*d*,2)-digraphs do exist for *any* degree. The digraph constructed is the line digraph of K_{d+1}, LK_{d+1}. Concerning the enumeration of (*d*,2)-digraphs, it is known from [9] that there are exactly three non-isomorphic (2,2)-digraphs (see Figure 1).



(1)(2)(3)(4)(5)(6) (a) (123)(456) (b) (1234)(56) (c)

Figure 1. The three non-isomorphic (2, 2)-digraphs

In [2], it is shown that there is exactly one (3,2)-digraph, i.e., LK_4. Fixing the degree instead of the diameter Miller and Fris [8] proved that (2,*k*)-digraphs do not exist for any values of $k \geq 3$. However, the existence problem of (*d*,*k*)-digraphs with $d \geq 3$ and $k \geq 3$ is still open.

Every (*d*,*k*)-digraph *G* has the characteristic property that for every vertex $x \in G$ there exists exactly one vertex *y* so that there are two walks of lengths $\leq k$ from *x* to *y* (one of them must be of length *k*). We called the vertex *y* is *the repeat* of *x* and denoted by *r*(*x*). If $r(x) = y$ then $r^{-1}(y) = x$. Thus the map $r : V(G) \to V(G)$ is a permutation on $V(G)$. If $r(x) = x$ then *x* is called *selfrepeat* (in this case, the two walks have lengths 0 and *k*). It means that *x* is contained in a C_k. If $r(x) \neq x$ then *x* is called *non-selfrepeat*. It is easy to show that no vertex of a (*d*,*k*)-digraphs is contained in two C_k's.

In this paper, we study the enumeration of (4,2)-digraphs. Particularly, we study (4,2)-digraphs containing a cycle of length 2.

The following theorem and lemma shown in [4] and [3] will be used in this paper repeatedly. Let *G* be a (*d*,*k*)-digraph and $S \subseteq V$. Let $r(S) = \{r(x) \mid x \in S\}$.

Theorem 1. *For every vertex v of a (d,k)-digraph, we have:*

(a) $N^+(r(v)) = r(N^+(v))$

(b) $N^-(r(v)) = r(N^-(v))$

In the other words, theorem 1 shows that $(a,b) \in G$ if and only if $(r(a), r(b)) \in G$.

Lemma 1. *The permutation r has the same cycle structure on $N^+(v)$ for every selfrepeat v of (d,k)-digraphs G.*

2. Results

The aim of this paper is to show that if a (4,2)-digraph contains a selfrepeat then *all* vertices in such a digraph must be selfrepeats.

Let *G* is a (4,2)-digraph that contains a selfrepeat vertex. We shall label the vertices of *G* by $0, 1, 2, \cdots, 19$. Without loss of generality, from now on we assume the following:

1. 0 is a selfrepeat vertex;
2. $N^+(0) = \{1,2,3,4\}$ and $(0,4) \in C_2$ (thus 4 is also a selfrepeat);
3. $N^+(1) = \{5,6,7,8\}$, $N^+(2) = \{9,10,11,12\}$, $N^+(3) = \{13,14,15,16\}$, and $N^+(4) = \{17,18,19,0\}$, (see figure 2).

We shall define $L_1 = \{1,2,3,4\}$, $L_1 = N^+(1) \cup N^+(2) \cup N^+(3) \cup N^+(4)$, and for each $i \in V(G)$, define $\Delta_i = \{i\} \cup N^+(i)$.

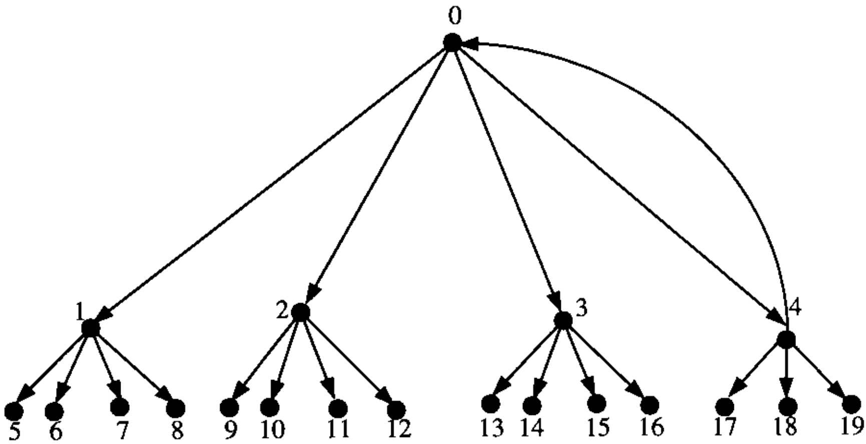


Figure 2. The (4,2)-digraphs with containing a cycle of length 2

Since 0 is a selfrepeat then for each $a \in L_1$, by Theorem 1, we have $r(a) \in L_1$. Furthermore, Theorem 1 implies that for each $b \in L_2$, we have $r(b) \in L_2$. Then we have following lemma.

Lemma 2. *For each* $j = 1,2$, *we have that if a* L_j, *then* $r(a) \in L_j$.

Lemma 3. *If x is a non-selfrepeat vertex in a (d,k)-digraph G and* $r(x) \in N^+(x)$ *then* $N^+(x)$ *does not contain any selfrepeat vertices.*

Proof. Consider any $y \in N^+(x)$. If $y = r(x)$ then y is a non-selfrepeat. Now, let $y \neq r(x)$. For a contradiction assumes that y is a selfrepeat. Since $(x,y) \in E(G)$, by Theorem 1 we have $(r(x), r(y) = y) \in E(G)$. Thus there are two walks of lengths ≤ 2 from x to y in G, namely (x,y) and $(x, r(x), y)$. Thus $r(x) = y$ which is not possible. Therefore, each vertex of $N^+(x)$ is a non-selfrepeat.

Lemma 4. *If x is a non-selfrepeat vertex in a (d,k)-digraph G and* $r(x) \in N^+(x)$ *then* $N^+(x)$ *does not contain any vertex and its repeat together.*

Proof. Suppose that vertex t and $r(f)$ are in $N^+(x)$. Since $(x,t) \in G$, due to Theorem 1, then we have $(r(x), r(t)) \in E(G)$. Thus there are two walks of lengths ≤ 2 from x to $r(t)$, namely $(x, r(t))$ and $(x, r(x), r(t))$. Thus $r(x) = r(t)$. Hence $x = t$, a contradiction with t in $N^+(x)$.

To show that each vertex in G is a selfrepeat. We consider the out-neighbors of 0. Since 0 and 4 are selfrepeats, then by Theorem 1 we essentially have three cases:

Case 1 Vertices 1, 2, and 3 are non-selfrepeat vertices.
Case 2 Two of $\{1,2,3\}$ are non-selfrepeat vertices.
Case 3 Vertices 1, 2, and 3 are selfrepeat vertices.

Let s be a selfrepeat in (4,2)-digraph G. Let t is a non-selfrepeat in $N^+(s)$. Then each vertex u in $N^+(t)$ must be a non-selfrepeat, since otherwise by Theorem 1 there are two walks from s to u which implies that $r(s) = u$, a contradiction with s being a selfrepeat. Let u be in $N^+(t)$. The following lemma considers the properties of out-neighbors of u.

Lemma 5. *Let s be a selfrepeat vertex in (4,2)-digraph G. Let $t \in N^+(s)$ be a non-selfrepeat vertex. Let $u \in N^+(t)$ be a non-selfrepeat vertex such that $(u,v) \in G$, for some $v \in N^+(s)$ and v is a non-selfrepeat vertex. Let $r(t) = v$. Then for each $y \in N^+(s)$, there is at most one non-selfrepeat w, where $w = N^+(u) \cap \Delta_y$.*

Proof. Suppose that there are two non-selfrepeat vertices of $N^+(u)$, which are in Δ_y, for some $y \in N^+(s)$. Since $r(t) = v$ and $(t,u) \in E(G)$, due to Theorem 1, then $(r(t) = v,\ r(u)) \in E(G)$. Hence $r(u)$ in $N^+(v)$. Suppose $N^+(u) = \{v, y_1, y_2, y_3\}$ and both y_1 and y_2 are in Δ_y. If one of them, say y_1, is equal to y, then there exist two walks of lengths ≤ 2 from u to y_2. This means that $r(u) = y_2$. Since $r(u)$ in $N^+(v)$, we should have an arc from v to y_2 in G. Thus; altogether there are three walks of lengths ≤ 2 from u to y_2, a contradiction. Thus, $y_1 \neq y$. Similarly, we can show that $y_2 \neq y$. Let us denote the three remaining vertices of Δ_y by y, x_1, and x_2 such that $N^+(y) = \{y_1, y_2, x_1, x_2\}$ (see Figure 4).

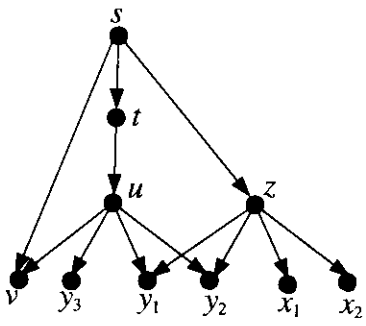


Figure 4

Of course $y \neq v$. Since otherwise, there are two repeats of u, namely $r(u) = y_1$ and $r(u) = y_2$. To reach y in 2 steps from u we cannot do via v, since there will be two walks of lengths ≤ 2 from s to y, namely (s, y) and (s, v, y). Thus $r(s) = y$, a contradiction with s being selfrepeat. We cannot do it via y_1 or y_2, since there will be a C_2 containing y_1 or y_2, a contradiction with y_1 or y_2 being a non-selfrepeat. Hence, to reach y from u we must do it through y_3. Thus we have $(y_3, y) \in E(G)$.

To reach x_1 in 2 steps from u we cannot do it via v, because if we have $(v, x_1) \in G$ then there are two walks of lengths ≤ 2 from s to x_1. Thus $r(s) = x_1$, a contradiction. We cannot do it through either y_1 or y_2, because if we have (y_1, x_1) or $(y_2, x_1) \in G$, then there are two walks of lengths ≤ 2 from y to x_1. Thus $r(y) = x_1$. Since s is a selfrepeat and $(s, y) \in G$, by Theorem 1, we have $(s, r(y) = x_1) \in G$. Thus there are also two walks of lengths ≤ 2 from s to x_1 in G, namely (s, y, x_1) and (s, x_1). Hence $r(s) = x_1$, a contradiction. Therefore, we have $(y_3, x_1) \in E(G)$ to be able to reach x_1 from u. Similarly, we can show to reach x_2 from u in 2 steps we should have $(y_3, x_2) \in E(G)$. Thus altogether implies $r(y_3) = x_1$ and x_2, a contradiction with the uniqueness of repeat. Therefore there are at most one out-neighbor of u which is in Δy.

In the following sections, we shall show that Cases 1 and 2 can not hold.

2.1. Case 1

Consider a (4,2)-digraph G containing a subdigraph of Figure 2 and having properties of Case 1. In this case, 1, 2, and 3 are non-selfrepeat vertices. Without loss of generality, we can assume that

$$r(1) = 2,\ r(2) = 3 \text{ and } r(3) = 1 \tag{3}$$

Then, we have the following three properties (due to Theorem 1):

1. if $a \in N^+(1)$ then $r(a) \in N^+(2)$,
2. if $a \in N^+(2)$ then $r(a) \in N^+(3)$,
3. If $a \in N^+(3)$ then $r(a) \in N^+(1)$.

Thus each vertex in $N^+(1) \cup N^+(2) \cup N^+(3)$ is a non-selfrepeat. Since $4 \in N^+(0)$ is a selfrepeat, then the permutation r on $N^+(4)$ has the same cycle structure with that on $N^+(0)$. In this case, $N^+(0)$ consists of three non-selfrepeat and one selfrepeat. Since $0 \in N^+(4)$ is selfrepeat, then $N^+(4) \setminus \{0\}$ consists of non-selfrepeat vertices.

Since G has diameter 2, hence to reach 1 from 3 there must exist a vertex $x_0 \in N^+(3)$ such that $(x_0, 1) \in G$. From now on, let us denote by x, y, and z the remaining three out-neighbors of x_0 in G. Of course, none of them can be 0 since otherwise $r(x_0) = 1$, a contradiction with $r(x_0)$ in $N^+(1)$. None of them can be in Δ_3. Since otherwise, then $r(3)$ in $N^+(3)$, a contradiction with assumption that $r(3) = 1$.

Lemma 6. *There is at most one of $\{x, y, z\}$ can be in either $N^+(1)$ or Δ_2, or $\Delta_4 \setminus \{0\}$.*

Proof. Suppose that two of $\{x, y, x\}$ be in $N^+(1)$, say x and y. Then $r(x_0) = x$ and y, a contradiction with the uniqueness of repeat. Hence at most one of $\{x, y, x\}$ be in $N^+(1)$. Suppose that two of $\{x, y, x\}$ be in Δ_2, say x and y. Since all of vertices in Δ_2 is non-selfrepeat, by Lemma 5 then at most one of x and y can be in Δ_2. Hence at most one of $\{x, y, x\}$ in Δ_2. Suppose that two of $\{x, y, x\}$ be in $\Delta_4 \setminus \{0\}$, say x and y. If one of them, say x, is equal to 4, then there exist two walks of lengths ≤ 2 from x_0 to y. Thus $r(x_0) = y \in N^+(4)$, a contradiction with $r(x_0) \in N^+(1)$. Thus $x \neq 4$. Similarly, we can show that $y \neq 4$. Hence both x and y be in $N^+(4) \setminus \{0\}$. Since all of vertices in $N^+(4) \setminus \{0\}$ is non-selfrepeat, by Lemma 5 then at most one of x and y can be in $N^+(4) \setminus \{0\}$. Hence at most one of $\{x, y, x\}$ in $\Delta_4 \setminus \{0\}$.

One of $\{x, y, x\}$ must be in $N^+(1)$. Since otherwise, then there are two of $\{x, y, x\}$ be in Δ_2 or $\Delta_4 \setminus \{0\}$, a contradiction with Lemma 6. Let x be in $N^+(1)$. Hence $r(x_0) = x \in N^+(x_0)$. Then y or z cannot be equal to 4. Since otherwise, then $N^+(x_0)$ contains a selfrepeat vertex, a contradiction with Lemma 3. Hence none of $\{y, z\}$ can be 4. If one of $\{y, z\}$ is equal to 2, then $N^+(x_0)$ contains 1 and $r(1) = 2$, a contradiction with Lemma 4.

The following theorem will complete the impossibility of case 1.

Theorem 2. *There is no (4,2)-digraph containing a subdigraph of Figure 2 and having properties of Case 1.*

Proof. Suppose that G is a (4,2)-digraph containing a subdigraph of Figure 2 and having properties of Case 1. By Lemma 6, we have that out-neighbors x, y, z of x_0 other than 1 must be equally distributed, namely $x \in N^+(1)$, $y \in N^+(2)$, and $z \in N^+(4) \setminus \{0\}$. Then $r(x_0) = x$. Since $r(x_0) = x$, $r(1) = 2$, and $(x_0, 1) \in G$, by using Theorem 1, then $(r(x_0),\ r(1)) = (x, 2) \in G$.

We will show that $(x, 3) \in G$. To reach 3 from x_0 in 2 steps, we cannot do this via 1, because $3 \neq N^+(1)$. If we do that via z, then there are two walks $(4, 0, 3)$ and $(4, z, 3)$ in G. Hence $r(4) = 3$ which is a contradiction with 4 being a selfrepeat. Suppose that $(y, 3) \in G$. Next, we must reach 0 from x_0 in 2 steps. We cannot do it via 1, because $0 \neq N^+(1)$. If we do that via x, then there are two walks from x to 2 or $r(x) = 2$, a contradiction with $r(x) \in N^+(2)$. If we do it via y, then $r(y) = 3$, a contradiction with $r(y) \in N^+(3)$. If we do that via z, then $r(4) = 0$, a contradiction with 4 is a selfrepeat. So, $(y, 3) \notin G$. This implies that $(x, 3) \in G$.

To reach 0 from x_0 in 2 steps, we cannot do this via 1, because $0 \neq N^+(1)$. If we do that via x, then there are two walks from x to 2. This means that $r(x) = 2$, a contradiction with $r(x) \in N^+(2)$. If we do that via z, then $r(4) = 0$, a contradiction with 4 is a selfrepeat. Hence $(y, 0)$ is in G. Similarly, to reach 4 from x_0 in 2 steps, we can show that it is done through x. Hence we have $(x, 4) \in G$.

Let t be the fourth vertex in $N^+(x)$. Now we consider vertex x and the others at distance 1 and 2 from x. At distance 1 from x, there are 2, 3, 4, and t. At distance 2 from x, $N^+(t)$ contain 1 and the remaining vertices in $N^+(1) \setminus \{x\}$ (since $N^+(2) = \{9, 10, 11, 12\}$, $N^+(3) = \{13, 14, 15, 16\}$, $N^+(4) = \{17, 18, 19, 0\}$). Then t has multiple repeats, a contradiction with the uniqueness repeat.

2.2. Case 2

Consider a (4,2)-digraph G containing a subdigraph of Figure 2 and having properties of case 2. In this case, there are two out-neighbors of 0 as non-selfrepeat vertices. Without loss of generality, we can assume that those non-selfrepeat vertices are 1, and 2, such that

$$r(1) = 2, \ r(2) = 1, \quad \text{and} \quad r(3) = 3 \tag{3}$$

Then, by Theorem 1 we have two following properties:

1. if $a \in N^+(1)$ then $r(a) \in N^+(2)$,
2. if $a \in N^+(2)$ then $r(a) \in N^+(1)$.

This implies that all vertices in $N^+(1) \cup N^+(2)$ are non-selfrepeat vertices. Since 3 and 4 are selfrepeats, then by Lemma 5, vertices 3 and 4 have the same cycle structure with 0. In this case, two of vertices in $N^+(0)$ are selfrepeat and the others are non-selfrepeat. Then $N^+(3)$ and $N^+(4)$ consist of two selfrepeat vertices and non-selfrepeat each. We assume that 15 and 16 are selfrepeat vertices in $N^+(3)$. Let $H_1 = \{15, 16\}$. It is clear 0 is a selfrepeat vertex in $N^+(4)$. Let another selfrepeat in $N^+(4)$ be 19. Let $H_2 = \{0, 19\}$. Since 3 is a selfrepeat then 3 contain in a C_2 which contain another selfrepeat vertex, say s. Then s only can be 15 or 16. Let s = 16. Hence 3 and 16 contain in a C_2. For 15 and 19, they must be containing in C_2. Since otherwise then there will be one of $\{0, 3, 4, 16\}$ contain in two cycle of length 2, a contradiction. Furthermore, since 15, 16, and 19 are selfrepeat vertices, then by Lemma 5, each of $N^+(15)$, $N^+(16)$, and $N^+(19)$ consist of two selfrepeat vertices and two non-selfrepeat.

Since G has diameter 2, hence to reach 1 from 2 there must exist a vertex $x_0 \in N^+(2)$ such that $(x_0, 1) \in G$. From now on, let us denote by x, y, and z the remaining three out-neighbors of x_0 in G. Of course, none of them can be 0 since otherwise $r(x_0) = 1$, a contradiction with $r(x_0)$ in $N^+(1)$. None of them can be in Δ_2. Since otherwise, then $r(2)$ in $N^+(2)$, a contradiction with assumption that $r(2) = 1$. If there are more than one of $\{x, y, z\}$ can be in Δ_3, then none of $\{x, y, z\}$ can be 3. Since otherwise, then there are two walks of lengths ≤ 2 from x_0 to a vertex in $N^+(3)$. Then $r(x_0) \in N^+(3)$, a contradiction with $r(x_0) \in N^+(1)$. Similarly, if there are more than one of $\{x, y, z\}$ can be in $\Delta_4 \setminus \{0\}$, then none of $\{x, y, z\}$ can be 4.

Proposition 1. $N^+(16) = N^+(0)$.

Proof. It is clear $3 \in N^+(16)$. Let $\in N^+(16)$ be $\{3, x_1, x_2, x_3\}$. Let x_1 be another selfrepeat vertex in $N^+(16)$. If $x_1 = 0$ then there are two walks of lengths ≤ 2 from 16 to 3, namely $(16, 0, 3)$ and $(16, 3)$. Thus $r(16) = 3$, a contradiction 16 being selfrepeat. Hence $x_1 \neq 0$. If $x_1 = 19$ then there are two walks of lengths ≤ 2 from 3 to 19, namely $(3, 15, 19)$ and $(3, 16, 19)$. Thus $r(3) = 19$, a contradiction with 3 being selfrepeat. Hence $x_1 \neq 19$. If $x_1 = 15$ then there are two walks of lengths ≤ 2 from 3 to 15, namely

$(3,16,15)$ and $(3,15)$. Thus $r(3)=15$, a contradiction with 3 being selfrepeat. Hence $x_1 \neq 15$. Hence $x_1 = 4$.

Vertex x_2 cannot contain in $N^+(3)\setminus\{16\}$ or $N^+(4)$, because if it can then $r(16)=x_2 \in N^+(3)\setminus\{16\}$ or $r(16)=x^2 \in N^+(0)$, a contradiction with 16 being selfrepeat vertex. Similarly, x_3 cannot be in $N^+(3)\setminus\{16\}$ or $N^+(4)$. Thus x_2 and x_3 must contain in Δ_1 and Δ_2.

Suppose that $x_2 \in N^+(1)$. Then, we consider vertex 16 and the others at distance 1 and 2 from 16. At distance 1 from 16, there are $3, 4, x_2$, and x_3. At distance 2 from 16, vertices of $N^+(x_2)$ cannot be 1 (since if they are, then there will be a C_2 contain 1) and vertices of $N^+(x_2)$ cannot be in $N^+(1)\setminus\{x_2\}$ (since if they are, then $r(1) \in N^+(1)$). Hence $N^+(x_2)$ will contain vertices in $\{2\} \cup N^+(2)$ (since $N^+(3)=\{13,14,15,16\}$ and $N^+(4)=\{0,17,18,19\}$). Then x_3 must be containing in $\{1,2\} \cup \{N^+(1)\setminus\{x_2\}\} \cup N^+(2)$. If $x_3 = 1$, then there are two walks of lengths ≤ 2 from 16 to x_2, namely $\{16, x_2\}$ and $\{16, 1, x_2\}$. Then $r(16)=x_2$, a contradiction with 16 being selfrepeat. If $x_3 = 2$, then at distance 2 from 16 there are $N^+(2)$, $N^+(3)$, $N^+(4)$, and $N^+(x_2)$. Thus $N^+(x_2)$ consists of $\{1\} \cup \{N^+(1)\setminus\{x_2\}\}$. Thus x_2 has *multiple repeats*, a contradiction with the uniqueness of repeat. If $x_3 \in \{N^+(1)\setminus\{x_2\}\}$, then 1 cannot be in $N^+(x_2)$ and $N^+(x_3)$. Thus 16 cannot reach 1 in a path of lengths ≤ 2, a *contradiction*. Hence $x_3 \notin \{N^+(1)\setminus\{x_2\}$. If $x_3 \in N^+(2)$, then $N^+(x_3)$ cannot contain 2 (if it can then there is a cycle contain 2, a contradiction). It means that 2 must be in $N^+(x_2)$. Then $N^+(x_2)$ consists of 2 and $\{N^+(2)\setminus\{x_3\}\}$. Thus x_2 has multiple repeat, a contradiction with the uniqueness of repeat. Then x_3 cannot be containing in $\{1,2\} \cup \{N^+(1)\setminus\{x_2\}\} \cup N^+(2)$, a contradiction. Hence x_2 cannot be in $N^+(1)$. Similarly x_2 cannot be in $N^+(2)$. Hence x_2 must be 1 or 2. Let $x_2 = 2$. Since 16 is a selfrepeat and $(16,2) \in E(G)$, by using Theorem 1, then $(r(16)=16,\ r(2)=1)\ E(G)$. Hence x_3 must be 1. Hence $N^+(16)=\{1,2,3,4\}$ $= N^+(0)$.

All of $\{x,y,z\}$ cannot be in Δ_3. Since otherwise, x_0 cannot reach the fourth vertex in Δ_3, say t (because we cannot do it via 1 and if we do it via one of $\{x,y,z\}$, say x, then there will be two walks of lengths ≤ 2 from 3 to x, a *contradiction*). As we know before that none of $\{x,y,z\}$ which are in Δ_3 can be 3. Hence there are at most two of $\{x,y,z\}$ can be in $N^+(3)$. Similarly, there are at most two of $\{x,y,z\}$ can be in $N^+(0)\setminus\{0\}$.

Lemma 7. *There is at most one of {x,y,z} can be in either $N^{+}(1)$ or $N^{+}(3)$ or $N^{+}(4)\backslash\{0\}$.*

Proof. Suppose that two of $\{x, y, z\}$ can be in $N^+(1)$, say x and y. Then $r(x_0) = x$ and y, a contradiction with the uniqueness of repeat. Hence there is at most one of $\{x,y,z\}$ can be in $N^+(1)$. Suppose that two of $\{x, y, z\}$ can be in $N^+(3)$, say x and y. Both x and y cannot be non-selfrepeat vertices. Since if they are then it will be a contradiction with Lemma 5. Hence both of $\{x, y\}$ is selfrepeat or $\{x, y\}$ consist of one selfrepeat and one non-selfrepeat. One of $\{x, y\}$ cannot be 16. Since otherwise, then there are two walks of lengths ≤ 2 from x_0 to 1 (because $N^+(16) = \{1, 2, 3, 4\}$). Then $r(x_0) = 1$, a contradiction with $r(x_0)$ in $N^+(1)$. Hence one of $\{x, y\}$ is equal to 15 and another is 13 and 14.

For $x = 13$ and $y = 15$. If $z = 19$, then there are two walks of lengths ≤ 2 from x_0 to 19, namely $(x_0, 19)$ and $(x_0, 15, 19)$ (because $19 \in N^+(15)$). Then $r(x_0) = 19$, a contradiction with $r(x_0)$ in $N^+(1)$. Hence $z \neq 19$. If $z = 4$, then there are two walks of lengths ≤ 2 from x_0 to 19, namely $(x_0, 15, 19)$ and $(x_0, 4, 19)$ (because $19 \in N^+(15)$ and $19 \in N^+(4)$). Then $r(x_0) = 19$, a contradiction. Suppose that $z = 18$. Then we consider x_0 and the others at distance 1 and 2 from x_0. At distance 1, we have 1, 13, 15, and 18. At distance 2, we have $N^+(1) = \{5, 6, 7, 8\}$, $N^+(13)$, $N^+(15)$, and $N^+(18)$. Now we consider where we can put 3. $N^+(13)$ cannot contain 3. Since otherwise, there will be a cycle of length 2 contain 13, a contradiction with 13 being a non-selfrepeat. $N^+(15)$ cannot contain 3. Since otherwise, then 3 in two C_2's, a contradiction. Hence $3 \in N^+(18)$. Now, we consider where we can put 16. $N^+(13)$ cannot contain 16. Since otherwise, then $r(3) = 16$, a contradiction with 3 being a selfrepeat. Similarly, $N^+(15)$ cannot contain 16. If $N^+(18)$ contain 16, then $r(18) = 16$, a contradiction with 16 being a selfrepeat. Thus we cannot reach 16 from x_0 in 1 and 2 steps, a contradiction. Similarly, if $z = 17$, we cannot reach 16 from x_0 in 1 and 2 steps. Thus z must be in $N^+(3)$. Hence all of $\{x, y, z\}$ must be in Δ_3, a contradiction. Similarly, for $y = 14$ and $z = 15$, then all of $\{x, y, z\}$ must be in Δ_3, a contradiction. Hence two of $\{x, y, z\}$ cannot be in $N^+(3)$. Similar reason we use to find a contradiction if two of $\{x, y, z\}$ can be in $N^+(4)\backslash\{0\}$. Thus two of $\{x, y, z\}$ cannot be in $N^+(4)\backslash\{0\}$. Hence there is at most one of $\{x,y,z\}$ can be in either $N^+(1)$ or $N^+(3)$ or $N^+(4)\backslash\{0\}$.

One of $\{x, y, z\}$ must be in $N^+(1)$. Since otherwise, then there are two of $\{x, y, z\}$ be in $N^+(3)$ or $N^+(4)\backslash\{0\}$, a contradiction with Lemma 7. Let x be in $N^+(1)$. Hence $r(x_0) = x \in N^+(x_0)$. Then y or z cannot contain in union of $\{4, 3\} \cup H_1 \cup H_2$. Since otherwise, then $N^+(x_0)$ contains a selfrepeat vertex, a contradiction with Lemma 3. Hence none of $\{y,z\}$ can contain in union of $\{4, 3\} \cup H_1 \cup H_2$.

Theorem 3. *There is no (4,2)-digraph containing a subdigraph of Figure 2 and having properties of Case 2.*

Proof. Suppose that G is a (4,2)-digraph containing a subdigraph of Figure 2 and having properties of Case 2. Due to Lemma 7, we have that the three out-neighbors x, y and z of x_0 other than 1 must be equally distributed, namely $x \in N^+(1)$, $y \in N^+(3)\backslash H_1$, and $z \in N^+(4)\backslash H_2$. Since $r(x_0) = x$, $r(1) = 2$, and $(x_0, 1) \in G$, by using Theorem 1, then $(r(x_0), r(1)) = (x, 2) \in G$. To reach 0 from x_0, we must do it from y, because if we do so via x or z then $r(x) = 2$ or $r(4) = 0$, a contradiction. Hence $(y, 0) \in G$. To reach 3 from x_0, we must do it through x, because if we do via y or z then 3 in two C_2's or $r(y) = 4$, respectively, a contradiction. Similarly, if we show that 4 is reachable from x_0 through x. Hence $(x, 3)$ and $(x, 4)$ are in G.

Let t be the remaining vertex in $N^+(x)$. Similarly with the proof of Theorem 2, we have multiple repeats for t, a contradiction with the uniqueness of repeat.

2.3. Case 3

Consider a (4,2)-digraph containing a subdigraph of Figure 2 and having properties of Case 3. In this case, we have that all out-neighbors of 0 are selfrepeats. We will complete our proof by showing that (4,2)-digraph is exactly LK_5.

Theorem 4. *There is exactly one (4,2)-digraph, which contains a selfrepeat, namely the line digraph LK_5 of complete digraph on 5 vertices.*

Proof. Since all out-neighbors of 0 are selfrepeats then by using Lemma 1 implies that all vertices in the digraph must be selfrepeats. Next, due to Theorem 3 in [4], we conclude that only such (4,2)-digraph is LK_5.

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