On \((4,2)\)-digraphs Containing a Cycle of Length 2

Hazrul Iswadi and Edy Tri Baskoro

Department of Mathematics and Sciences, University of Surabaya, Jalan Raya Kali Rungkut, Surabaya 60292, Indonesia

Department of Mathematics, Institut Teknologi Bandung, Jalan Ganesha 10, Bandung, Indonesia

d-mail: us6179@ubaya.ac.id
d-mail: itbma0l@bdg.centrin.net.id

Abstract A digraph is a digraph with the in-degree and out-degree of all vertices is constant. The Moore bound for a digraph of degree \(d\) and diameter \(k\) is \(M_{d,k} = 1 + d + d^2 + \ldots + d^k\). It is well known that digraphs of order \(M_{d,k}\), degree \(d > 1\) and diameter \(k > 1\) do not exist. A \((d,k)\)-digraph is a digraph of degree \(d > 1\), diameter \(k > 1\), and number of vertices one less than the Moore bound. For degrees \(d = 2\) and 3, it has been shown that for diameter \(k \geq 3\) there are no such \((d,k)\)-digraphs. However for diameter 2, it is known that \((d,2)\)-digraphs do exist for any degree \(d\). The line digraph of \(K_{d+1}\) is one example of such \((d,2)\)-digraphs. Furthermore, the recent study showed that there are three non-isomorphic \((2,2)\)-digraphs and exactly one non-isomorphic \((3,2)\)-digraph. In this paper, we shall study \((4,2)\)-digraphs. We show that if \((4,2)\)-digraph \(G\) contains a cycle of length 2 then \(G\) must be the line digraph of a complete digraph \(K_4\).

1. Introduction

A digraph \(G\) is a system consisting of a finite nonempty set \(V(G)\) of objects called vertices and a set \(E(G)\) of ordered pairs of distinct vertices called arcs. The order of \(G\) is the cardinality of \(V(G)\). A subdigraph \(H\) of \(G\) is a digraph having all vertices and arcs in \(G\). If \((u,v)\) is an arc in a digraph \(G\), then \(u\) is said to be adjacent to \(v\) and \(v\) is said to be adjacent from \(u\). An in-neighbor of a vertex \(v\) in a digraph \(G\) is a vertex \(u\) such that \((u,v) \in G\). An out-neighbor of a vertex \(v\) in a digraph \(G\) is a vertex \(w\) such that \((v,w) \in G\). The set of all out-neighbors of a vertex \(v\) is denoted by \(N^+(v)\) and its cardinality is called the out-degree of \(v\), \(d^+(v) = |N^+(v)|\). Similarly, the set of all in-neighbors of a vertex \(v\) is denoted by \(N^-(v)\) and its cardinality is called the in-degree of \(v\), \(d^-(v) = |N^-(v)|\). A digraph \(G\) is digraphic of degree \(d\) if for any vertex \(v\) in \(G\), \(d^+(v) = d^-(v) = d\).

A walk of length \(h\) from a vertex \(u\) to vertex \(v\) in \(G\) is a sequence of vertices \((u = u_0, u_1, \ldots, u_h = v)\) such that \((u_{i-1}, u_i) \in G\) for each \(i\). A vertex \(u\) forms the trivial
walk of length 0. A closed walk has \( u_0 = u_h \). A path is a walk in which all points are distinct. A cycle \( C_h \) of length \( h > 0 \) is a closed walk with \( h \) distinct vertices (except \( u_0 \) and \( u_h \)). If there is a path from \( u \) to \( v \) in \( G \) then we say that \( v \) is reachable from \( u \).

The distance from vertex \( u \) to vertex \( v \) in a digraph \( G \), denoted by \( \delta(u, v) \), is defined as the length of a shortest path from \( u \) to \( v \). In general, \( \delta(u, v) \) is not necessarily equal to \( \delta(v, u) \). The diameter \( k \) of a digraph \( G \) is the maximum distance between any two vertices in \( G \).

Let \( G \) be a digraph of degree \( d \) and diameter \( k \) with \( n \) vertices. Let one vertex be distinguished in \( G \). Let \( n_i, \forall i = 0, 1, \ldots, k \), be the number of vertices at distance \( i \) from the distinguished vertex. Then,

\[
n_i \leq d^i, \text{ for } i = 1, \ldots, k.
\]

Hence,

\[
n = \sum_{i=0}^{k} n_i \leq 1 + d + d^2 + \cdots + d^k.
\]

The number of \( 1 + d + d^2 + \cdots + d^k \) is the upper bound for the number of vertices in digraph \( G \). This upper bound is called Moore bound and denoted by \( M_{d,k} \). If the equality sign in (2) holds then the digraph \( G \) is called Moore digraph.

It has been known that the Moore digraphs do not exist for \( d > 1 \) and \( k > 1 \), except for trivial cases (for \( d = 1 \) or \( k = 1 \)), [10] and [5]. The trivial cases are fulfilled by the cycle digraph \( C_{k+1} \) for \( d = 1 \), and the complete digraph \( K_{d+1} \) for \( k = 1 \). This motivates the study of the existence problem of digraphs of degree \( d \), diameter \( k \) with order \( M_{d,k} - 1 \). Such digraphs are called Almost Moore digraphs and denoted by \( (d,k) \)-digraphs.

Several results have been obtained on the existence of \( (d,k) \)-digraphs. For instance, in [6] it is shown that the \( (d,2) \)-digraphs do exist for any degree. The digraph constructed is the line digraph of \( K_{d+1} \), \( LK_{d+1} \). Concerning the enumeration of \( (d,2) \)-digraphs, it is known from [9] that there are exactly three non-isomorphic \( (2,2) \)-digraphs (see Figure 1).

![Figure 1. The three non-isomorphic (2, 2)-digraphs](image-url)
In [2], it is shown that there is exactly one \((3,2)\)-digraph, i.e., \(LK_4\). Fixing the degree instead of the diameter Miller and Fris [8] proved that \((2,k)\)-digraphs do not exist for any values of \(k \geq 3\). However, the existence problem of \((d,k)\)-digraphs with \(d \geq 3\) and \(k \geq 3\) is still open.

Every \((d,k)\)-digraph \(G\) has the characteristic property that for every vertex \(x \in G\) there exists exactly one vertex \(y\) so that there are two walks of lengths \(\leq k\) from \(x\) to \(y\) (one of them must be of length \(k\)). We called the vertex \(y\) is the repeat of \(x\) and denoted by \(r(x)\). If \(r(x) = y\) then \(r^{-1}(y) = x\). Thus the map \(r: V(G) \rightarrow V(G)\) is a permutation on \(V(G)\). If \(r(x) = x\) then \(x\) is called selfrepeat (in this case, the two walks have lengths 0 and \(k\)). It means that \(x\) is contained in a \(C_k\). If \(r(x) \neq x\) then \(x\) is called non-selfrepeat. It is easy to show that no vertex of a \((d,k)\)-digraphs is contained in two \(C_k\)'s.

In this paper, we study the enumeration of \((4,2)\)-digraphs. Particularly, we study \((4,2)\)-digraphs containing a cycle of length 2.

The following theorem and lemma shown in [4] and [3] will be used in this paper repeatedly. Let \(G\) be a \((d,r(-)\)-digraph and \(S \subseteq V\). Let \(r(S) = \{r(x) | x \in S\}\).

**Theorem 1.** For every vertex \(v\) of a \((d,k)\)-digraph, we have:

\[
\begin{align*}
(a) \quad N^+(r(v)) &= r(N^+(v)) \\
(b) \quad N^-(r(v)) &= r(N^-(v))
\end{align*}
\]

In the other words, theorem 1 shows that \((a,b) \in G\) if and only if \((r(a),r(b)) \in G\).

**Lemma 1.** The permutation \(r\) has the same cycle structure on \(N^+(v)\) for every selfrepeat \(v\) of \((d,k)\)-digraphs \(G\).

### 2. Results

The aim of this paper is to show that if a \((4,2)\)-digraph contains a selfrepeat then all vertices in such a digraph must be selfrepeats.

Let \(G\) is a \((4,2)\)-digraph that contains a selfrepeat vertex. We shall label the vertices of \(G\) by 0, 1, 2, ..., 19. Without loss of generality, from now on we assume the following:

1. 0 is a selfrepeat vertex;
2. \(N^+(0) = \{1, 2, 3, 4\}\) and \((0, 4) \in C_2\) (thus 4 is also a selfrepeat);
3. \(N^+(1) = \{5, 6, 7, 8\}\), \(N^+(2) = \{9, 10, 11, 12\}\), \(N^+(3) = \{13, 14, 15, 16\}\), and \(N^+(4) = \{17, 18, 19, 0\}\). (see figure 2).
We shall define $L_i = \{1, 2, 3, 4\}$, $L_i = N^+(1) \cup N^+(2) \cup N^+(3) \cup N^+(4)$, and for each $i \in V(G)$, define $\Delta_i = \{i\} \cup N^+(i)$.

Since 0 is a self-repeat then for each $a \in L_1$, by Theorem 1, we have $r(a) \in L_1$. Furthermore, Theorem 1 implies that for each $b \in L_2$, we have $r(b) \in L_2$. Then we have following lemma.

**Lemma 2.** For each $j=1,2$, we have that if $a \in L_j$, then $r(a) \in L_j$.

**Lemma 3.** If $x$ is a non-selfrepeat vertex in a $(d,k)$-digraph $G$ and $r(x) \in N^+(x)$ then $N^+(x)$ does not contain any selfrepeat vertices.

**Proof.** Consider any $y \in N^+(x)$. If $y = r(x)$ then $y$ is a non-selfrepeat. Now, let $y \neq r(x)$. For a contradiction assumes that $y$ is a selfrepeat. Since $(x, y) \in E(G)$, by Theorem 1 we have $(r(x), r(y)) = (y) \in E(G)$. Thus there are two walks of lengths $\leq 2$ from $x$ to $y$ in $G$, namely $(x, y)$ and $(x, r(x), y)$. Thus $r(x) = y$ which is not possible. Therefore, each vertex of $N^+(x)$ is a non-selfrepeat.

**Lemma 4.** If $x$ is a non-selfrepeat vertex in a $(d,k)$-digraph $G$ and $r(x) \in N^+(x)$ then $N^+(x)$ does not contain any vertex and its repeat together.
Proof. Suppose that vertex $t$ and $r(f)$ are in $N^+(s)$. Since $(x,t) \in E(G)$, due to Theorem 1, then we have $(r(x), r(t)) \in E(G)$. Thus there are two walks of lengths $\leq 2$ from $x$ to $r(t)$, namely $(x, r(t))$ and $(x, r(x), r(t))$. Thus $r(x) = r(t)$. Hence $x = t$, a contradiction with $t$ in $N^+(x)$.

To show that each vertex in $G$ is a selfrepeat. We consider the out-neighbors of $s$. Since $0$ and $4$ are selfrepeats, then by Theorem 1 we essentially have three cases:

Case 1 Vertices $1$, $2$, and $3$ are non-selfrepeat vertices.
Case 2 Two of $\{1, 2, 3\}$ are non-selfrepeat vertices.
Case 3 Vertices $1$, $2$, and $3$ are selfrepeat vertices.

Let $s$ be a selfrepeat in $(4,2)$-digraph $G$. Let $t$ be a non-selfrepeat in $N^+(s)$. Then each vertex $u$ in $N^+(t)$ must be a non-selfrepeat, since otherwise by Theorem 1 there are two walks from $s$ to $u$ which implies that $r(s) = u$, a contradiction with $s$ being a selfrepeat.

Lemma 5. Let $s$ be a selfrepeat vertex in $(4,2)$-digraph $G$. Let $t \in N^+(s)$ be a non-selfrepeat vertex. Let $u \in N^+(t)$ be a non-selfrepeat vertex such that $(u, v) \in E(G)$, for some $v \in N^+(s)$ and $v$ is a non-selfrepeat vertex. Let $r(t) = v$. Then for each $y \in N^+(s)$, there is at most one non-selfrepeat $w$, where $w = N^+(u) \cap \Delta_y$.

Proof. Suppose that there are two non-selfrepeat vertices of $N^+(u)$, which are in $\Delta_y$, for some $y \in N^+(s)$. Since $r(t) = v$ and $(t, u) \in E(G)$, due to Theorem 1, then $(r(t) = v, r(u)) \in E(G)$. Hence $r(u)$ in $N^+(v)$. Suppose $N^+(u) = \{v, y_1, y_2, y_3\}$ and both $y_1$ and $y_2$ are in $\Delta_y$. If one of them, say $y_1$, is equal to $y$, then there exist two walks of lengths $\leq 2$ from $u$ to $y_2$. This means that $r(u) = y_2$. Since $r(u)$ in $N^+(v)$, we should have an arc from $v$ to $y_2$ in $G$. Thus, altogether there are three walks of lengths $\leq 2$ from $u$ to $y_2$, a contradiction. Thus, $y_1 \neq y$. Similarly, we can show that $y_2 \neq y$. Let us denote the three remaining vertices of $\Delta_y$ by $y, x_1$, and $x_2$ such that $N^+(y) = \{y_1, y_2, x_1, x_2\}$ (see Figure 4).
Of course \( y \neq v \). Since otherwise, there are two repeats of \( u \), namely \( r(u) = y_1 \) and \( r(u) = y_2 \). To reach \( y \) in 2 steps from \( u \) we cannot do via \( v \), since there will be two walks of lengths \( \leq 2 \) from \( s \) to \( y \), namely \( (s, y) \) and \( (s, v, y) \). Thus \( r(s) = y \), a contradiction with \( s \) being selfrepeat. We cannot do it via \( y_1 \) or \( y_2 \), since there will be a \( C_2 \) containing \( y_1 \) or \( y_2 \), a contradiction with \( y_1 \) or \( y_2 \) being a non-selfrepeat. Hence, to reach \( y \) from \( u \) we must do it through \( y_3 \). Thus we have \( (y_3, y) \in E(G) \).

To reach \( x_1 \) in 2 steps from \( u \) we cannot do it via \( v \), because if we have \( (v, x_1) \in G \) then there are two walks of lengths \( \leq 2 \) from \( s \) to \( x_1 \). Thus \( r(s) = x_1 \), a contradiction. We cannot do it through either \( y_1 \) or \( y_2 \), because if we have \( (y_1, x_1) \) or \( (y_2, x_1) \in G \), then there are two walks of lengths \( \leq 2 \) from \( y \) to \( x_1 \). Thus \( r(y) = x_1 \). Since \( s \) is a selfrepeat and \( (s, y) \in G \), by Theorem 1, we have \( (s, r(y) = x_1) \in G \). Thus there are also two walks of lengths \( \leq 2 \) from \( s \) to \( x_1 \) in \( G \), namely \( (s, y, x_1) \) and \( (s, x_1) \). Hence \( r(s) = x_1 \), a contradiction. Therefore, we have \( (y_3, x_1) \in E(G) \) to be able to reach \( x_1 \) from \( u \). Similarly, we can show to reach \( x_2 \) from \( u \) in 2 steps we should have \( (y_3, x_2) \in E(G) \). Thus altogether implies \( r(y_3) = x_1 \) and \( x_2 \), a contradiction with the uniqueness of repeat. Therefore there are at most one out-neighbor of \( u \) which is in \( \Delta y \).

In the following sections, we shall show that Cases 1 and 2 can not hold.

2.1. Case 1

Consider a \((4,2)\)-digraph \( G \) containing a subdigraph of Figure 2 and having properties of Case 1. In this case, 1, 2, and 3 are non-selfrepeat vertices. Without loss of generality, we can assume that

\[
 r(1) = 2, \quad r(2) = 3 \quad \text{and} \quad r(3) = 1
\]  \( (3) \)
Then, we have the following three properties (due to Theorem 1):

1. If $a \in N^+(l)$ then $r(a) \in N^+(2)$,
2. If $a \in N^+(2)$ then $r(a) \in N^+(3)$,
3. If $a \in N^+(3)$ then $r(a) \in N^+(l)$.

Thus each vertex in $N^+(1) \cup N^+(2) \cup N^+(3)$ is a non-selfrepeat. Since $4 \in N^+(0)$ is a selfrepeat, then the permutation $r$ on $N^+(4)$ has the same cycle structure with that on $N^+(0)$. In this case, $N^+(0)$ consists of three non-selfrepeat and one selfrepeat. Since $0 \in N^+(4)$ is selfrepeat, then $N^+(4) \setminus \{0\}$ consists of non-selfrepeat vertices.

Since $G$ has diameter 2, hence to reach 1 from 3 there must exist a vertex $x_0 \in N^+(3)$ such that $(x_0, 1) \in G$. From now on, let us denote by $x, y,$ and $z$ the remaining three out-neighbors of $x_0$ in $G$. Of course, none of them can be 0 since otherwise $r(x_0) = 1$, a contradiction with $r(x_0) \in N^+(1)$. None of them can be in $A_4$, since otherwise, then $r(3)$ in $N^+(3)$, a contradiction with assumption that $r(3) = 1$.

**Lemma 6.** There is at most one of $\{x, y, z\}$ can be in either $N^+(1)$ or $A_2$, or $A_4 \setminus \{0\}$.

**Proof.** Suppose that two of $(x, y, z)$ be in $N^+(1)$, say $x$ and $y$. Then $r(x_0) = x$ and $y$, a contradiction with the uniqueness of repeat. Hence at most one of $\{x, y, x\}$ be in $N^+(1)$. Suppose that two of $(x, y, x)$ be in $A_2$, say $x$ and $y$. Since all of vertices in $A_2$ is non-selfrepeat, by Lemma 5 then at most one of $x$ and $y$ can be in $A_2$. Hence at most one of $\{x, y, x\}$ in $A_2$. Suppose that two of $(x, y, x)$ be in $A_4 \setminus \{0\}$, say $x$ and $y$. If one of them, say $x$, is equal to 4, then there exist two walks of lengths $\leq 2$ from $x_0$ to $y$. Thus $r(x_0) = y \in N^+(4)$, a contradiction with $r(x_0) \in N^+(l)$. Thus $x \neq 4$. Similarly, we can show that $y \neq 4$. Hence both $x$ and $y$ be in $N^+(4) \setminus \{0\}$. Since all of vertices in $N^+(4) \setminus \{0\}$ is non-selfrepeat, by Lemma 5 then at most one of $x$ and $y$ be in $N^+(4) \setminus \{0\}$. Hence at most one of $(x, y, x)$ in $A_4 \setminus \{0\}$.

One of $(x, y, x)$ must be in $N^+(1)$. Since otherwise, then there are two of $(x, y, x)$ be in $A_2$ or $A_4 \setminus \{0\}$, a contradiction with Lemma 6. Let $x$ be in $N^+(1)$. Hence $r(x_0) = x \in N^+(x_0)$. Then $y$ or $z$ cannot be equal to 4. Since otherwise, then $N^+(x_0)$ contains a selfrepeat vertex, a contradiction with Lemma 3. Hence none of $(y, z)$ can
be 4. If one of \( \{y, z\} \) is equal to 2, then \( N^+(x_0) \) contains 1 and \( r(1) = 2 \), a contradiction with Lemma 4.

The following theorem will complete the impossibility of case 1.

**Theorem 2.** There is no \((4,2)\)-digraph containing a subdigraph of Figure 2 and having properties of Case 1.

**Proof.** Suppose that \( G \) is a \((4,2)\)-digraph containing a subdigraph of Figure 2 and having properties of Case 1. By Lemma 6, we have that out-neighbors \( x, y, z \) of \( x_0 \) other than 1 must be equally distributed, namely \( x \in N^+(1) \), \( y \in N^+(2) \), and \( z \in N^+(4) \setminus \{0\} \). Then \( r(x_0) = x \). Since \( r(x_0) = x \), \( r(1) = 2 \), and \( (x_0, 1) \in G \), by using Theorem 1, then \( (r(x_0), r(1)) = (x, 2) \in G \).

We will show that \( (x, 3) \in G \). To reach 3 from \( x_0 \) in 2 steps, we cannot do this via 1, because \( 3 \neq N^+(1) \). If we do that via \( z \), then there are two walks \((4, 0, 3)\) and \((4, z, 3)\) in \( G \). Hence \( r(4) = 3 \) which is a contradiction with 4 being a self-repeat. Suppose that \( (y, 3) \in G \). Next, we must reach 0 from \( x_0 \) in 2 steps. We cannot do it via 1, because \( 0 \neq N^+(1) \). If we do that via \( x \), then there are two walks from \( x \) to 2 or \( r(x) = 2 \), a contradiction with \( r(x) \in N^+(2) \). If we do it via \( y \), then \( r(y) = 3 \), a contradiction with \( r(y) \in N^+(3) \). If we do that via \( z \), then \( r(4) = 0 \), a contradiction with 4 is a self-repeat. So, \( (y, 3) \notin G \). This implies that \( (x, 3) \in G \).

To reach 0 from \( x_0 \) in 2 steps, we cannot do this via 1, because \( 0 \neq N^+(1) \). If we do that via \( x \), then there are two walks from \( x \) to 2. This means that \( r(x) = 2 \), a contradiction with \( r(x) \in N^+(2) \). If we do that via \( z \), then \( r(4) = 0 \), a contradiction with 4 is a self-repeat. Hence \( (y, 0) \) is in \( G \). Similarly, to reach 4 from \( x_0 \) in 2 steps, we can show that it is done through \( x \). Hence we have \( (x, 4) \in G \).

Let \( t \) be the fourth vertex in \( N^+(x) \). Now we consider vertex \( x \) and the others at distance 1 and 2 from \( x \). At distance 1 from \( x \), there are 2, 3, 4, and \( t \). At distance 2 from \( x \), \( N^+(t) \) contain 1 and the remaining vertices in \( N^+(1) \setminus \{x\} \) (since \( N^+(2) = \{9, 10, 11, 12\} \), \( N^+(3) = \{13, 14, 15, 16\} \), \( N^+(4) = \{17, 18, 19, 0\} \) ). Then \( t \) has multiple repeats, a contradiction with the uniqueness repeat.

**2.2. Case 2**

Consider a \((4,2)\)-digraph \( G \) containing a subdigraph of Figure 2 and having properties of case 2. In this case, there are two out-neighbors of 0 as non-self-repeat vertices. Without loss of generality, we can assume that those non-self-repeat vertices are 1, and 2, such that

\[
\begin{align*}
\text{r}(1) &= 2, \\
\text{r}(2) &= 1, \\
\text{r}(3) &= 3
\end{align*}
\]
Then, by Theorem 1 we have two following properties:

1. if \( a \in N^+(1) \) then \( \sigma(a) \in N^+(2) \),
2. if \( a \in N^+(2) \) then \( \sigma(a) \in N^+(1) \).

This implies that all vertices in \( N^+(1) \cup N^+(2) \) are non-selfrepeat vertices. Since 3 and 4 are self-repeats, then by Lemma 5, vertices 3 and 4 have the same cycle structure with 0. In this case, two of vertices in \( N^+(0) \) are selfrepeat and the others are non-selfrepeat. Then \( N^+(3) \) and \( N^+(4) \) consist of two selfrepeat vertices and non-selfrepeat each.

We assume that 15 and 16 are selfrepeat vertices in \( N^+(3) \). Let \( H_1 = \{15, 16\} \). It is clear 0 is a selfrepeat vertex in \( N^+(4) \). Let another selfrepeat in \( N^+(4) \) be 19. Let \( H_2 = \{0, 19\} \). Since 3 is a selfrepeat then 3 contain in a \( C_2 \) which contain another selfrepeat vertex, say \( s \). Then \( s \) only can be 15 or 16. Let \( s = 16 \). Hence 3 and 16 contain in a \( C_2 \). For 15 and 19, they must be containing in \( C_2 \). Since otherwise then there will be one of \( 0, 3, 4, 16 \) contain in two cycle of length 2, a contradiction. Furthermore, since 15, 16, and 19 are selfrepeat vertices, then by Lemma 5, each of \( N^-(15) \), \( N^-(16) \), and \( N^-(19) \) consist of two selfrepeat vertices and two non-selfrepeat.

Since \( G \) has diameter 2, hence to reach 1 from 2 there must exist a vertex \( x_0 \in N^+(2) \) such that \((x_0, 1) \in E(G)\). From now on, let us denote by \( x, y, \) and \( z \) the remaining three out-neighbors of \( x_0 \) in \( G \). Of course, none of them can be 0 since otherwise \( r(x_0) = 1 \), a contradiction with \( r(x_0) \in N^+(1) \). None of them can be in \( \Delta_2 \). Since otherwise, then \( r(2) \) in \( N^+(2) \), a contradiction with assumption that \( r(2) = 1 \). If there are more than one of \( \{x, y, z\} \) can be in \( \Delta_3 \), then none of \( \{x, y, z\} \) can be 3. Since otherwise, then there are two walks of lengths \( \leq 2 \) from \( x_0 \) to a vertex in \( N^+(3) \). Then \( r(x_0) \in N^+(3) \), a contradiction with \( r(x_0) \in N^+(1) \). Similarly, if there are more than one of \( \{x, y, z\} \) can be in \( \Delta_4 \setminus \{0\} \), then none of \( \{x, y, z\} \) can be 4.

**Proposition 1.** \( N^+(16) = N^+(0) \).

**Proof.** It is clear \( 3 \in N^+(16) \). Let \( E \in N^+(16) \) be \( \{3, x_1, x_2, x_3\} \). Let \( x_1 \) be another selfrepeat vertex in \( N^+(16) \). If \( x_1 = 0 \) then there are two walks of lengths \( \leq 2 \) from 16 to 3, namely \((16, 0, 3)\) and \((16, 3)\). Thus \( r(16) = 3 \), a contradiction 16 being selfrepeat. Hence \( x_1 \neq 0 \). If \( x_1 = 19 \) then there are two walks of lengths \( \leq 2 \) from 3 to 19, namely \((3, 15, 19)\) and \((3, 16, 19)\). Thus \( r(3) = 19 \), a contradiction with 3 being selfrepeat. Hence \( x_1 \neq 19 \). If \( x_1 = 15 \) then there are two walks of lengths \( \leq 2 \) from 3 to 15, namely
(3, 16, 15) and (3, 15). Thus $r(3)=15$, a contradiction with 3 being selfrepeat. Hence $x_1 \neq 15$. Hence $x_1 = 4$.

Vertex $x_2$ cannot contain in $N^+(3) \setminus \{16\}$ or $N^+(4)$, because if it can then $r(16) = x_2 \in N^+(3) \setminus \{16\}$ or $r(16) = x^2 \in N^+(0)$, a contradiction with 16 being selfrepeat vertex. Similarly, $x_3$ cannot be in $N^+(3) \setminus \{16\}$ or $N^+(4)$. Thus $x_2$ and $x_3$ must contain in $\Delta_1$ and $\Delta_2$.

Suppose that $x_2 \in N^+(1)$. Then, we consider vertex 16 and the others at distance 1 and 2 from 16. At distance 1 from 16, there are 3, 4, $x_2$, and $x_3$. At distance 2 from 16, vertices of $N^+(x_2)$ cannot be 1 (since if they are, then there will be a $C_2$ contain 1) and vertices of $N^+(x_2)$ cannot be in $N^+(1) \setminus \{x_2\}$ (since if they are, then $r(1) \in N^+(1)$). Hence $N^+(x_2)$ will contain vertices in $\{2\} \cup N^+(2)$ (since $N^+(3) = \{13, 14, 15, 16\}$ and $N^+(4) = \{0, 17, 18, 19\}$). Then $x_3$ must be containing in $\{1, 2\} \cup \{N^+(1) \setminus \{x_2\}\} \cup N^+(2)$. If $x_3 = 1$, then there are two walks of lengths ≤ 2 from 16 to $x_2$, namely $\{16, x_2\}$ and $\{16, 1, x_2\}$. Then $r(16) = x_2$, a contradiction with 16 being selfrepeat. If $x_3 = 2$, then at distance 2 from 16 there are $N^+(2), N^+(3), N^+(4), \text{and } N^+(x_2)$. Thus $N^+(x_2)$ consists of $\{1\} \cup \{N^+(1) \setminus \{x_2\}\}$. Thus $x_2$ has multiple repeat, a contradiction with the uniqueness of repeat. If $x_3 \in \{N^+(1) \setminus \{x_2\}\}$, then 1 cannot be in $N^+(x_2)$ and $N^+(x_3)$. Thus 16 cannot reach 1 in a path of lengths ≤ 2, a contradiction. Hence $x_3 \notin \{N^+(1) \setminus \{x_2\}\}$. If $x_3 \in N^+(2)$, then $N^+(x_3)$ cannot contain 2 (if it can then there is a cycle contain 2, a contradiction). It means that 2 must be in $N^+(x_2)$. Then $N^+(x_2)$ consists of 2 and $\{N^+(2) \setminus \{x_3\}\}$. Thus $x_2$ has multiple repeat, a contradiction with the uniqueness of repeat. Then $x_3$ cannot be containing in $\{1, 2\} \cup \{N^+(1) \setminus \{x_2\}\} \cup N^+(2)$, a contradiction. Hence $x_2$ cannot be in $N^+(1)$. Similarly $x_2$ cannot be in $N^+(2)$. Hence $x_2$ must be 1 or 2. Let $x_2 = 2$. Since 16 is a selfrepeat and $(16, 2) \in E(G)$, by using Theorem 1, then $(r(16) = 16, r(2) = 1) \in E(G)$. Hence $x_3$ must be 1. Hence $N^+(16) = \{1, 2, 3, 4\} = N^+(0)$.

All of $\{x, y, z\}$ cannot be in $\Delta_3$. Since otherwise, $x_0$ cannot reach the fourth vertex in $\Delta_3$, say $t$ (because we cannot do it via 1 and if we do it via one of $\{x, y, z\}$, say $x$, then there will be two walks of lengths ≤ 2 from 3 to $x$, a contradiction). As we know before that none of $\{x, y, z\}$ which are in $\Delta_3$ can be 3. Hence there are at most two of $\{x, y, z\}$ can be in $N^+(3)$. Similarly, there are at most two of $\{x, y, z\}$ can be in $N^+(0) \setminus \{0\}$. 
Lemma 7. There is at most one of \( \{x, y, z\} \) can be in either \( N^+(1) \) or \( N^+(3) \) or \( N^+(4) \) \( \backslash \{0\} \).

Proof. Suppose that two of \( \{x, y, z\} \) can be in \( N^+(1) \), say \( x \) and \( y \). Then \( r(x_0) = x \) and \( y \), a contradiction with the uniqueness of repeat. Hence there is at most one of \( \{x, y, z\} \) can be in \( N^+(1) \). Suppose that two of \( \{x, y, z\} \) can be in \( N^+(3) \), say \( x \) and \( y \). Both \( x \) and \( y \) cannot be non-selfrepeat vertices. Since if they are then it will be a contradiction with Lemma 5. Hence both of \( \{x, y\} \) is selfrepeat or \( \{x, y\} \) consist of one selfrepeat and one non-selfrepeat. One of \( \{x, y\} \) cannot be 16. Since otherwise, then there are two walks of lengths \( \leq 2 \) from \( x_0 \) to 1 (because \( N^+(16) = \{1, 2, 3, 4\} \)). Then \( r(x_0) = 1 \), a contradiction with \( r(x_0) \) in \( N^+(1) \). Hence one of \( \{x, y\} \) is equal to 15 and another is 13 and 14.

For \( x = 13 \) and \( y = 15 \). If \( z = 19 \), then there are two walks of lengths \( \leq 2 \) from \( x_0 \) to 19, namely \( (x_0, 19) \) and \( (x_0, 15, 19) \) (because \( 19 \in N^+(15) \)). Then \( r(x_0) = 19 \), a contradiction with \( r(x_0) \) in \( N^+(1) \). Hence \( z \neq 19 \). If \( z = 4 \), then there are two walks of lengths \( \leq 2 \) from \( x_0 \) to 19, namely \( (x_0, 15, 19) \) and \( (x_0, 4, 19) \) (because \( 19 \in N^+(15) \) and \( 19 \in N^+(4) \)). Then \( r(x_0) = 19 \), a contradiction. Suppose that \( z = 18 \). Then we consider \( x_0 \) and the others at distance 1 and 2 from \( x_0 \). At distance 1, we have 1, 13, 15, and 18. At distance 2, we have \( N^+(1) = \{5, 6, 7, 8\} \), \( N^+(13) \), \( N^+(15) \), and \( N^+(18) \). Now we consider where we can put 3. \( N^+(13) \) cannot contain 3. Since otherwise, there will be a cycle of length 2 contain 13, a contradiction with 13 being a non-selfrepeat. \( N^+(15) \) cannot contain 3. Since otherwise, then 3 in two \( C_2 \)'s, a contradiction. Hence \( 3 \in N^+(18) \). Now, we consider where we can put 16. \( N^+(13) \) cannot contain 16. Since otherwise, then \( r(3) = 16 \), a contradiction with 3 being a selfrepeat. Similarly, \( N^+(15) \) cannot contain 16. If \( N^+(18) \) contain 16, then \( r(18) = 16 \), a contradiction with 16 being a selfrepeat. Thus we cannot reach 16 from \( x_0 \) in 1 and 2 steps, a contradiction. Similarly, if \( z = 17 \), we cannot reach 16 from \( x_0 \) in 1 and 2 steps. Thus \( z \) must be in \( N^+(3) \). Hence all of \( \{x, y, z\} \) must be in \( \Delta_3 \), a contradiction. Similarly, for \( y = 14 \) and \( z = 15 \), then all of \( \{x, y, z\} \) must be in \( \Delta_3 \), a contradiction. Hence two of \( \{x, y, z\} \) cannot be in \( N^+(3) \). Similar reason we use to find a contradiction if two of \( \{x, y, z\} \) can be in \( N^+(4) \) \( \backslash \{0\} \). Thus two of \( \{x, y, z\} \) cannot be in \( N^+(4) \) \( \backslash \{0\} \). Hence there is at most one of \( \{x, y, z\} \) can be in either \( N^+(1) \) or \( N^+(3) \) or \( N^+(4) \) \( \backslash \{0\} \).
One of \{x, y, z\} must be in \(N^+(1)\). Since otherwise, then there are two of \{x, y, z\} be in \(N^+(3)\) or \(N^+(4)\setminus\{0\}\), a contradiction with Lemma 7. Let \(x\) be in \(N^+(1)\). Hence \(r(x_0) = x \in N^+(x_0)\). Then \(y\) or \(z\) cannot contain in union of \(\{4, 3\} \cup H_1 \cup H_2\). Since otherwise, then \(N^+(x_0)\) contains a selfrepeat vertex, a contradiction with Lemma 3. Hence none of \{y, z\} can contain in union of \(\{4, 3\} \cup H_1 \cup H_2\).

**Theorem 3.** There is no (4,2)-digraph containing a subdigraph of Figure 2 and having properties of Case 2.

**Proof.** Suppose that \(G\) is a (4,2)-digraph containing a subdigraph of Figure 2 and having properties of Case 2. Due to Lemma 7, we have that the three out-neighbors \(x, y\) and \(z\) of \(x_0\) other than 1 must be equally distributed, namely \(x \in N^+(1)\), \(y \in N^+(3) \setminus H_1\), and \(z \in N^+(4) \setminus H_2\). Since \(r(x_0) = x\), \(r(1) = 2\), and \((x_0, 1) \in G\), by using Theorem 1, then \((r(x_0), r(1)) = (x, 2) \in G\). To reach 0 from \(x_0\), we must do it from \(y\), because if we do so via \(x\) or \(z\) then \(r(x) = 2\) or \(r(4) = 0\), a contradiction. Hence \((y, 0) \in G\). To reach 3 from \(x_0\), we must do it through \(x\), because if we do via \(y\) or \(z\) then 3 in two \(C_2\)'s or \(r(y) = 4\), respectively, a contradiction. Similarly, if we show that 4 is reachable from \(x_0\) through \(x\). Hence \((x, 3)\) and \((x, 4)\) are in \(G\).

Let \(r\) be the remaining vertex in \(N^+(x)\). Similarly with the proof of Theorem 2, we have multiple repeats for \(r\), a contradiction with the uniqueness of repeat.

**2.3. Case 3**

Consider a (4,2)-digraph containing a subdigraph of Figure 2 and having properties of Case 3. In this case, we have that all out-neighbors of 0 are selfrepeats. We will complete our proof by showing that (4,2)-digraph is exactly \(LK_5\).

**Theorem 4.** There is exactly one (4,2)-digraph, which contains a selfrepeat, namely the line digraph \(LK_5\) of complete digraph on 5 vertices.

**Proof.** Since all out-neighbors of 0 are selfrepeats then by using Lemma 1 implies that all vertices in the digraph must be selfrepeats. Next, due to Theorem 3 in [4], we conclude that only such (4,2)-digraph is \(LK_5\).

**Acknowledgement.** This research was partially supported by TWAS Research Grant No. 98-286 RG/MATH/AS.
On (4,2)-digraphs Containing a Cycle of Length

References


3. E.T. Baskoro, M. Miller, J. Siran, and M. Sutton, Complete characterization of almost Moore digraphs of degree three, to appear in *J. Graph Theory*.


