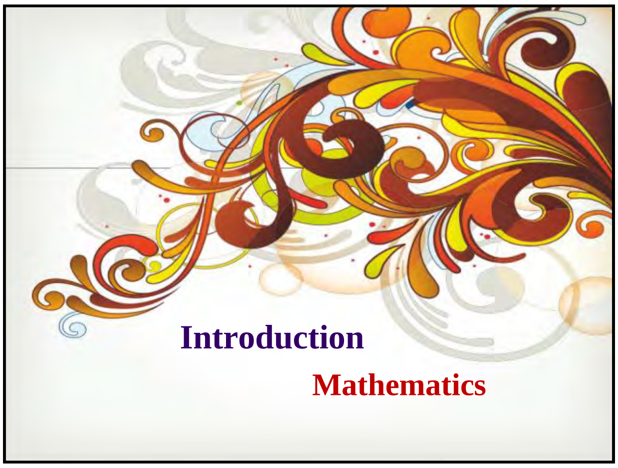
1st Half Semester Content

Mathematics

Odd Semester 2016 - 2017

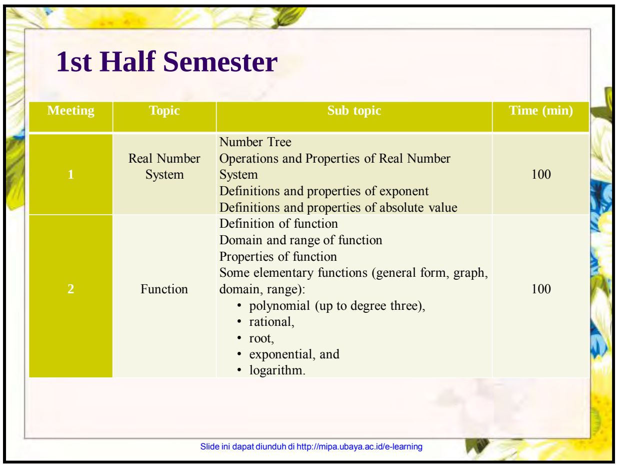
Lecturer: Hazrul Iswadi

Content



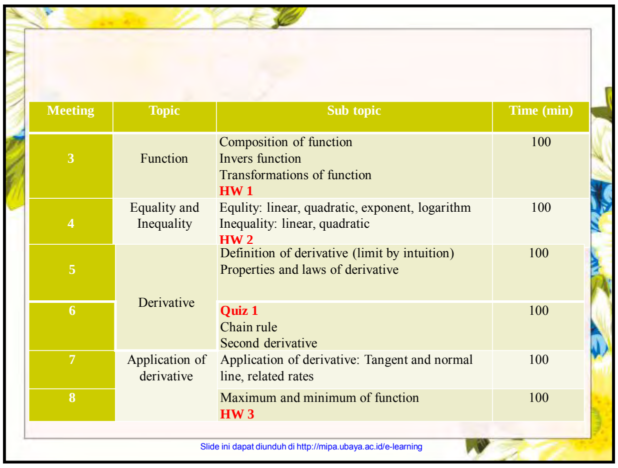
Introduction

Mathematics

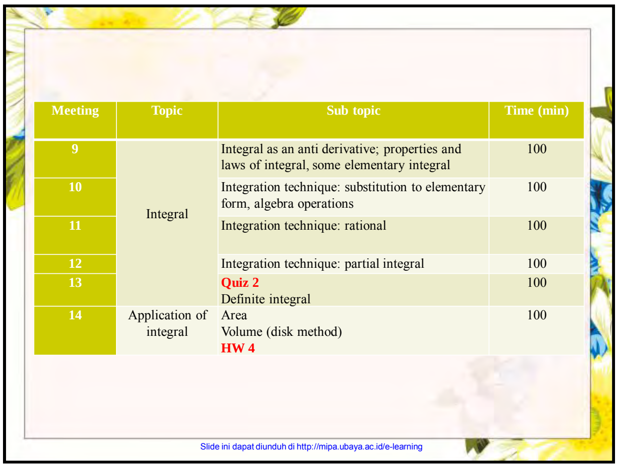


1st Half Semester

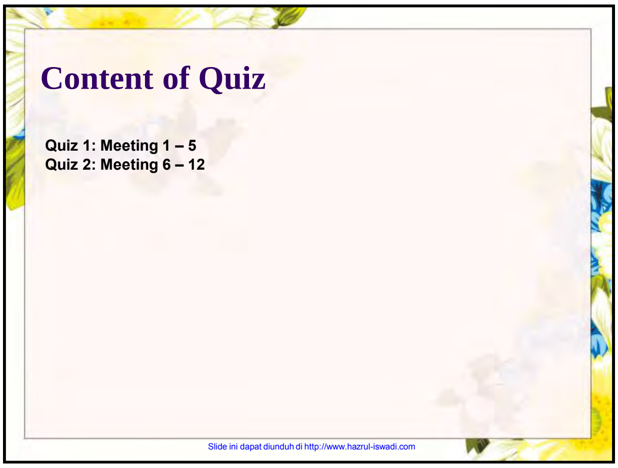
Meeting	Topic	Sub topic	Time (min)
1	Real Number System	Number Tree Operations and Properties of Real Number System Definitions and properties of exponent Definitions and properties of absolute value	100
2	Function	Definition of function Domain and range of function Properties of function Some elementary functions (general form, graph, domain, range): • polynomial (up to degree three), • rational, • root, • exponential, and • logarithm.	100



Meeting	Topic	Sub topic	Time (min)
3	Function	Composition of function Invers function Transformations of function **HW 1**	100
4	Equality and Inequality	Equlity: linear, quadratic, exponent, logarithm Inequality: linear, quadratic **HW 2**	100
5	Derivative	Definition of derivative (limit by intuition) Properties and laws of derivative	100
6		**Quiz 1** Chain rule Second derivative	100
7	Application of derivative	Application of derivative: Tangent and normal line, related rates	100
8		Maximum and minimum of function **HW 3**	100

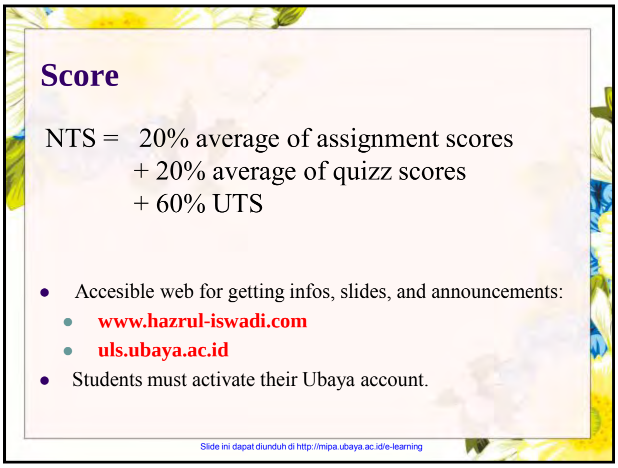


Meeting	Topic	Sub topic	Time (min)
9	Integral	Integral as an anti derivative; properties and laws of integral, some elementary integral	100
10		Integration technique: substitution to elementary form, algebra operations	100
11		Integration technique: rational	100
12		Integration technique: partial integral	100
13		**Quiz 2** Definite integral	100
14	Application of integral	Area Volume (disk method) **HW 4**	100



Content of Quiz

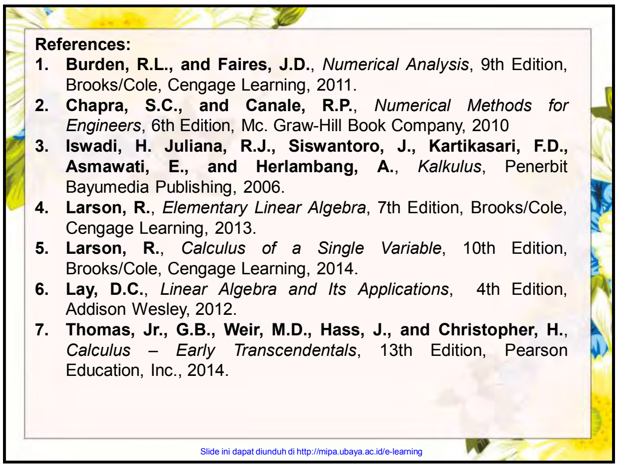
Quiz 1: Meeting 1 – 5
Quiz 2: Meeting 6 – 12



Score

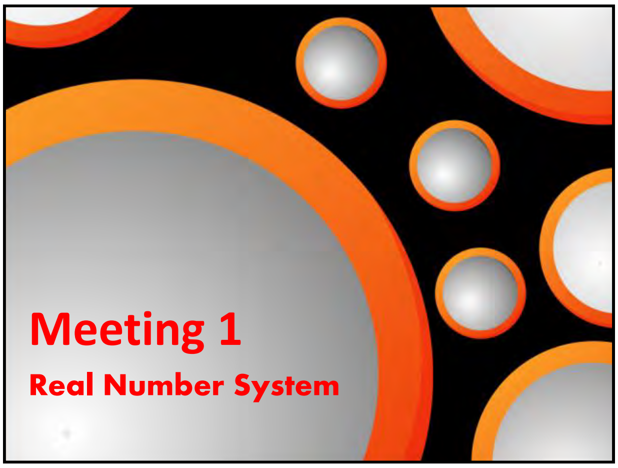
NTS = 20% average of assignment scores
+ 20% average of quizz scores
+ 60% UTS

- Accesible web for getting infos, slides, and announcements:
  - **www.hazrul-iswadi.com**
  - **uls.ubaya.ac.id**
- Students must activate their Ubaya account.



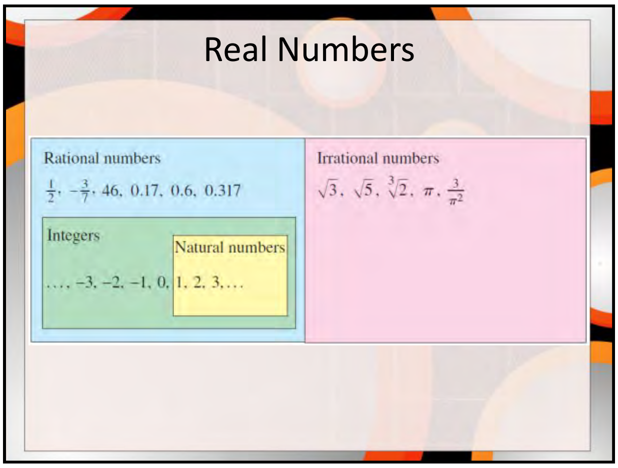
References:

1. **Burden, R.L., and Faires, J.D.**, *Numerical Analysis*, 9th Edition, Brooks/Cole, Cengage Learning, 2011.
2. **Chapra, S.C., and Canale, R.P.**, *Numerical Methods for Engineers*, 6th Edition, Mc. Graw-Hill Book Company, 2010
3. **Iswadi, H. Juliana, R.J., Siswantoro, J., Kartikasari, F.D., Asmawati, E., and Herlambang, A.**, *Kalkulus*, Penerbit Bayumedia Publishing, 2006.
4. **Larson, R.**, *Elementary Linear Algebra*, 7th Edition, Brooks/Cole, Cengage Learning, 2013.
5. **Larson, R.**, *Calculus of a Single Variable*, 10th Edition, Brooks/Cole, Cengage Learning, 2014.
6. **Lay, D.C.**, *Linear Algebra and Its Applications*, 4th Edition, Addison Wesley, 2012.
7. **Thomas, Jr., G.B., Weir, M.D., Hass, J., and Christopher, H.**, *Calculus – Early Transcendentals*, 13th Edition, Pearson Education, Inc., 2014.



Meeting 1

Real Number System



Real Numbers

Rational numbers

$\frac{1}{2}, -\frac{3}{7}, 46, 0.17, 0.6, 0.317$

Integers

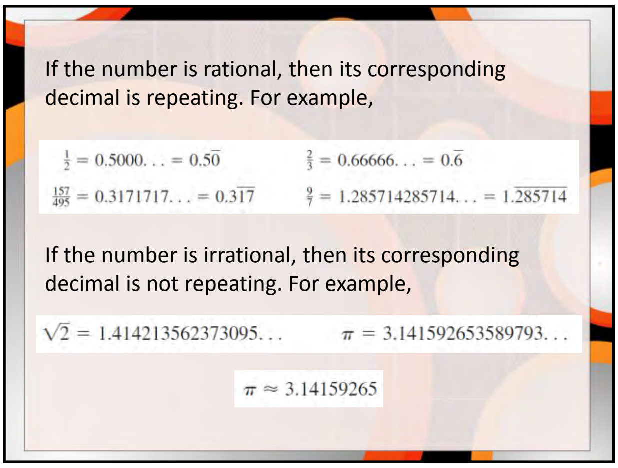
$\ldots, -3, -2, -1, 0,$

Natural numbers

$1, 2, 3, \ldots$

Irrational numbers

$\sqrt{3}, \sqrt{5}, \sqrt[3]{2}, \pi, \frac{3}{\pi^2}$



If the number is rational, then its corresponding decimal is repeating. For example,

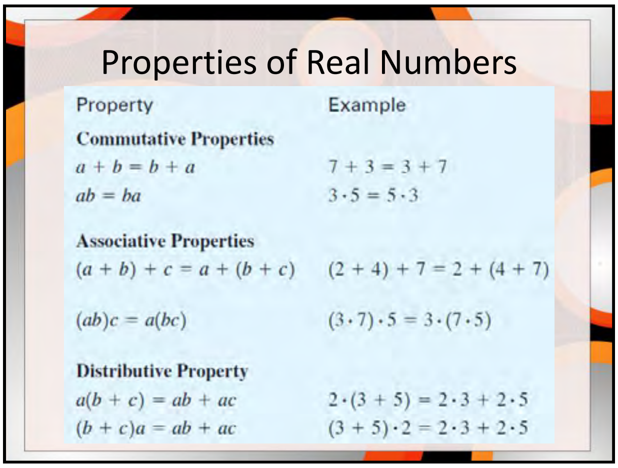
$\frac{1}{2} = 0.5000\ldots = 0.5\overline{0}$ $\qquad$ $\frac{2}{3} = 0.66666\ldots = 0.\overline{6}$

$\frac{157}{495} = 0.3171717\ldots = 0.3\overline{17}$ $\qquad$ $\frac{9}{7} = 1.285714285714\ldots = 1.\overline{285714}$

If the number is irrational, then its corresponding decimal is not repeating. For example,

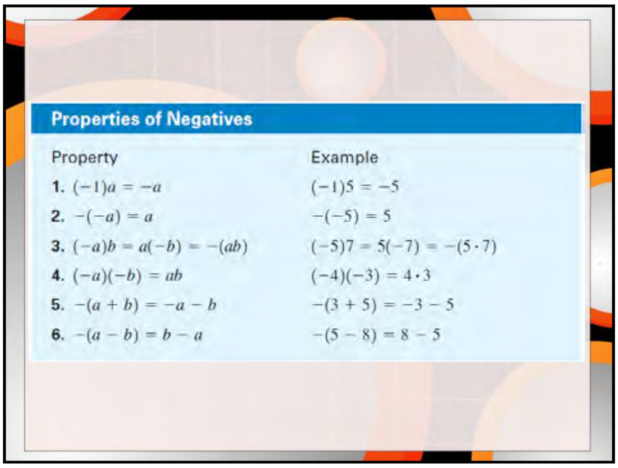
$\sqrt{2} = 1.414213562373095\ldots$ $\qquad$ $\pi = 3.141592653589793\ldots$

$\pi \approx 3.14159265$



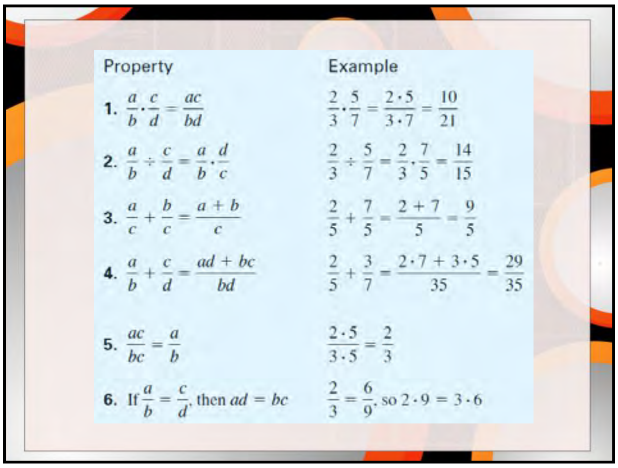
Properties of Real Numbers

Property	Example
Commutative Properties	
$a + b = b + a$	$7 + 3 = 3 + 7$
$ab = ba$	$3 \cdot 5 = 5 \cdot 3$
Associative Properties	
$(a + b) + c = a + (b + c)$	$(2 + 4) + 7 = 2 + (4 + 7)$
$(ab)c = a(bc)$	$(3 \cdot 7) \cdot 5 = 3 \cdot (7 \cdot 5)$
Distributive Property	
$a(b + c) = ab + ac$	$2 \cdot (3 + 5) = 2 \cdot 3 + 2 \cdot 5$
$(b + c)a = ab + ac$	$(3 + 5) \cdot 2 = 2 \cdot 3 + 2 \cdot 5$

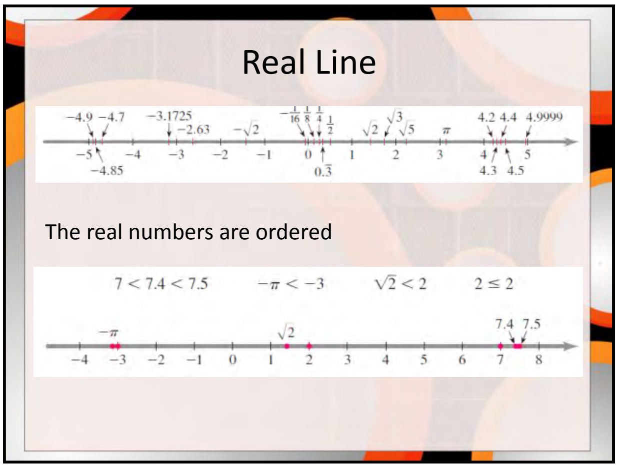


Properties of Negatives

Property	Example
1. $(-1)a = -a$	$(-1)5 = -5$
2. $-(-a) = a$	$-(-5) = 5$
3. $(-a)b = a(-b) = -(ab)$	$(-5)7 = 5(-7) = -(5 \cdot 7)$
4. $(-a)(-b) = ab$	$(-4)(-3) = 4 \cdot 3$
5. $-(a + b) = -a - b$	$-(3 + 5) = -3 - 5$
6. $-(a - b) = b - a$	$-(5 - 8) = 8 - 5$



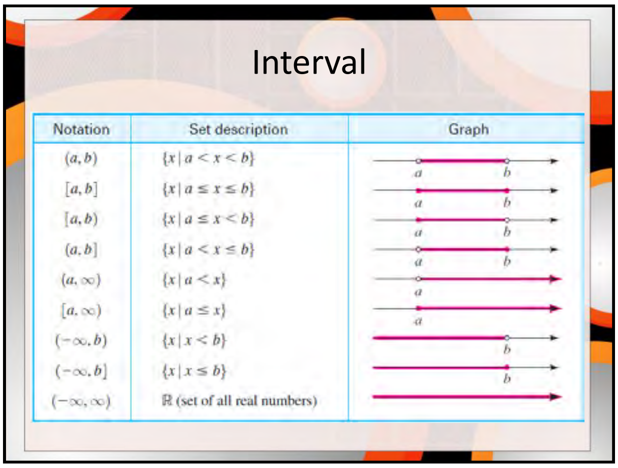
Property	Example
1. $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$	$\frac{2}{3} \cdot \frac{5}{7} = \frac{2 \cdot 5}{3 \cdot 7} = \frac{10}{21}$
2. $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$	$\frac{2}{3} \div \frac{5}{7} = \frac{2}{3} \cdot \frac{7}{5} = \frac{14}{15}$
3. $\frac{a}{c} + \frac{b}{c} = \frac{a + b}{c}$	$\frac{2}{5} + \frac{7}{5} = \frac{2 + 7}{5} = \frac{9}{5}$
4. $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$	$\frac{2}{5} + \frac{3}{7} = \frac{2 \cdot 7 + 3 \cdot 5}{35} = \frac{29}{35}$
5. $\frac{ac}{bc} = \frac{a}{b}$	$\frac{2 \cdot 5}{3 \cdot 5} = \frac{2}{3}$
6. If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$	$\frac{2}{3} = \frac{6}{9}$, so $2 \cdot 9 = 3 \cdot 6$



Real Line

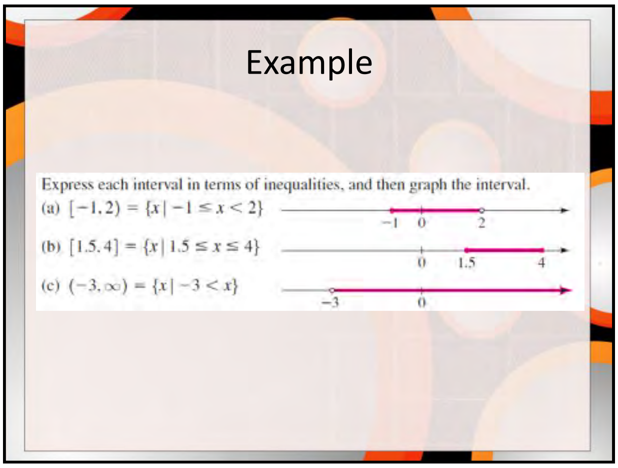
The real numbers are ordered

$7 < 7.4 < 7.5$ $-\pi < -3$ $\sqrt{2} < 2$ $2 \le 2$



Interval

Notation	Set description	Graph
(a, b)	$\{x \mid a < x < b\}$	a b
$[a, b]$	$\{x \mid a \le x \le b\}$	a b
$[a, b)$	$\{x \mid a \le x < b\}$	a b
$(a, b]$	$\{x \mid a < x \le b\}$	a b
(a, ∞)	$\{x \mid a < x\}$	a
$[a, \infty)$	$\{x \mid a \le x\}$	a
$(-\infty, b)$	$\{x \mid x < b\}$	b
$(-\infty, b]$	$\{x \mid x \le b\}$	b
$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	



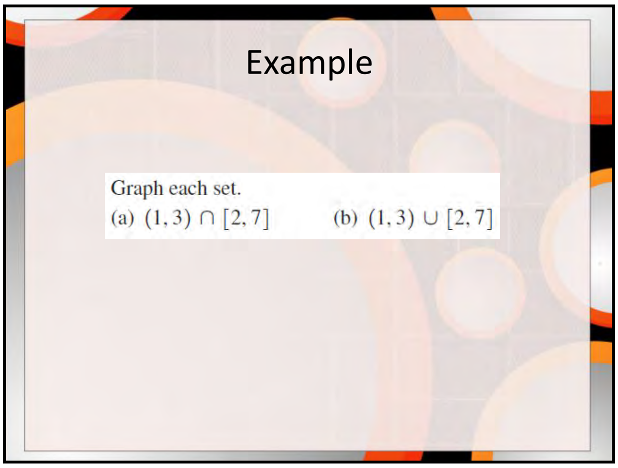
Example

Express each interval in terms of inequalities, and then graph the interval.

(a) $[-1, 2) = \{x \mid -1 \le x < 2\}$

(b) $[1.5, 4] = \{x \mid 1.5 \le x \le 4\}$

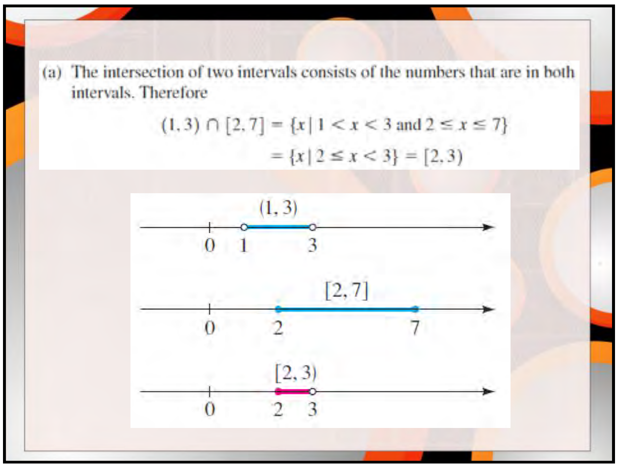
(c) $(-3, \infty) = \{x \mid -3 < x\}$



Example

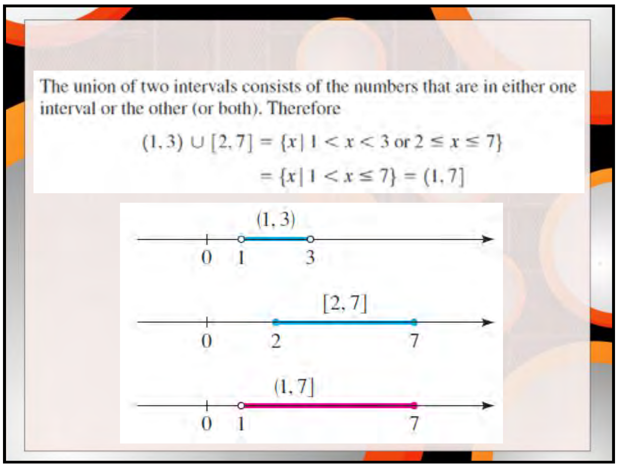
Graph each set.

(a) $(1, 3) \cap [2, 7]$ (b) $(1, 3) \cup [2, 7]$



(a) The intersection of two intervals consists of the numbers that are in both intervals. Therefore

$$
\begin{aligned}
(1, 3) \cap [2, 7] &= \{x \mid 1 < x < 3 \text{ and } 2 \le x \le 7\} \\
&= \{x \mid 2 \le x < 3\} = [2, 3)
\end{aligned}
$$



The union of two intervals consists of the numbers that are in either one interval or the other (or both). Therefore

$$
\begin{aligned}
(1, 3) \cup [2, 7] &= \{x \mid 1 < x < 3 \text{ or } 2 \le x \le 7\} \\
&= \{x \mid 1 < x \le 7\} = (1, 7]
\end{aligned}
$$

Exercises

1 – 6: Express the inequality in interval notation, and then graph the corresponding interval.

1. $x \le 1$
2. $-2 < x \le 1$
3. $1 \le x \le 2$
4. $x \ge -5$
5. $x > -1$
6. $5 < x < 2$

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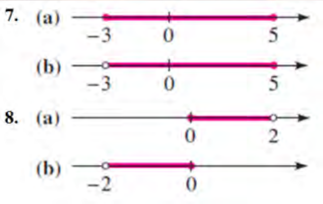
7 – 8: Express each set in interval notation.



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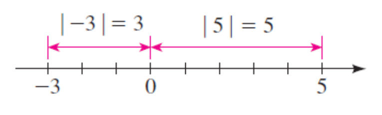
9 – 14: Graph the set.

9.	$(-2, 0) \cup (-1, 1)$	**10.**	$(-2, 0) \cap (-1, 1)$
11.	$[-4, 6] \cap [0, 8)$	**12.**	$[-4, 6) \cup [0, 8)$
13	$(-\infty, -4) \cup (4, \infty)$	**14.**	$(-\infty, 6] \cap (2, 10)$

Absolute Value and Distance

The **absolute value** of a number *a*, denoted by $|a|$, is the distance from *a* to 0 on the real number line.



Definition of Absolute Value

If *a* is a real number, then the **absolute value** of *a* is

$$|a| = \begin{cases} a & \text{if } a \ge 0 \\ -a & \text{if } a < 0 \end{cases}$$

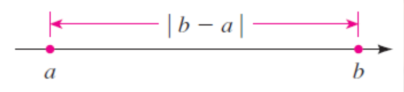
Properties of Absolute Value

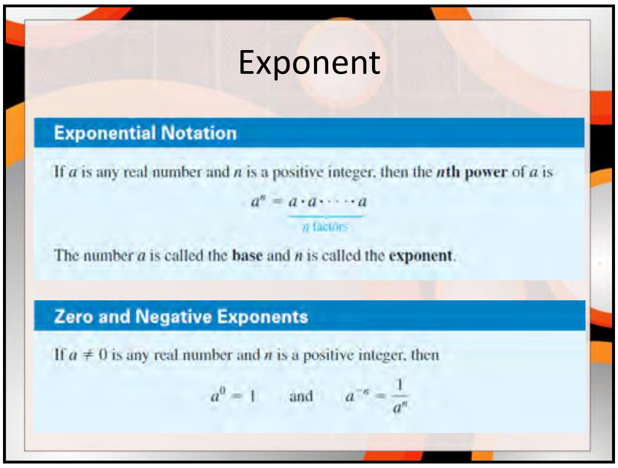
Property	Example
1. $\lvert a\rvert \ge 0$	$\lvert -3\rvert = 3 \ge 0$
2. $\lvert a\rvert = \lvert -a\rvert$	$\lvert 5\rvert = \lvert -5\rvert$
3. $\lvert ab\rvert = \lvert a\rvert\lvert b\rvert$	$\lvert -2\cdot 5\rvert = \lvert -2\rvert\lvert 5\rvert$
4. $\left\lvert \frac{a}{b}\right\rvert = \frac{\lvert a\rvert}{\lvert b\rvert}$	$\left\lvert \frac{12}{-3}\right\rvert = \frac{\lvert 12\rvert}{\lvert -3\rvert}$

Distance between Points on the Real Line

If *a* and *b* are real numbers, then the **distance** between the points *a* and *b* on the real line is

$$d(a, b) = |b - a|$$





Exponent

Exponential Notation

If a is any real number and n is a positive integer, then the **nth power** of a is

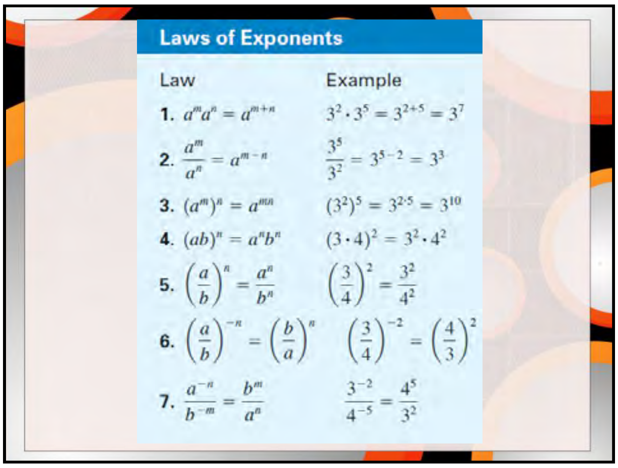
$$a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}}$$

The number a is called the **base** and n is called the **exponent**.

Zero and Negative Exponents

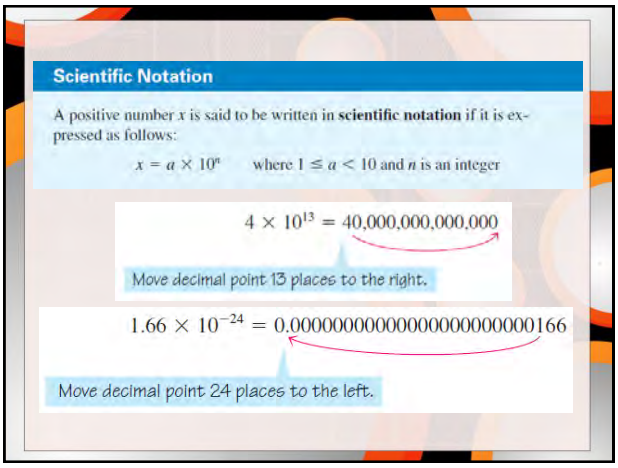
If $a \neq 0$ is any real number and n is a positive integer, then

$$a^0 = 1 \qquad \text{and} \qquad a^{-n} = \frac{1}{a^n}$$



Laws of Exponents

Law	Example
1. $a^m a^n = a^{m+n}$	$3^2 \cdot 3^5 = 3^{2+5} = 3^7$
2. $\frac{a^m}{a^n} = a^{m-n}$	$\frac{3^5}{3^2} = 3^{5-2} = 3^3$
3. $(a^m)^n = a^{mn}$	$(3^2)^5 = 3^{2 \cdot 5} = 3^{10}$
4. $(ab)^n = a^n b^n$	$(3 \cdot 4)^2 = 3^2 \cdot 4^2$
5. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	$\left(\frac{3}{4}\right)^2 = \frac{3^2}{4^2}$
6. $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$	$\left(\frac{3}{4}\right)^{-2} = \left(\frac{4}{3}\right)^2$
7. $\frac{a^{-n}}{b^{-m}} = \frac{b^m}{a^n}$	$\frac{3^{-2}}{4^{-5}} = \frac{4^5}{3^2}$



Scientific Notation

A positive number x is said to be written in **scientific notation** if it is expressed as follows:

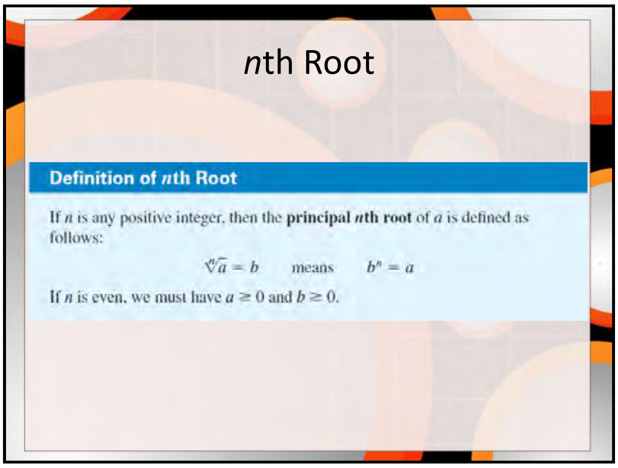
$$x = a \times 10^n \qquad \text{where } 1 \leq a < 10 \text{ and } n \text{ is an integer}$$

$$4 \times 10^{13} = 40{,}000{,}000{,}000{,}000$$

Move decimal point 13 places to the right.

$$1.66 \times 10^{-24} = 0.00000000000000000000000166$$

Move decimal point 24 places to the left.



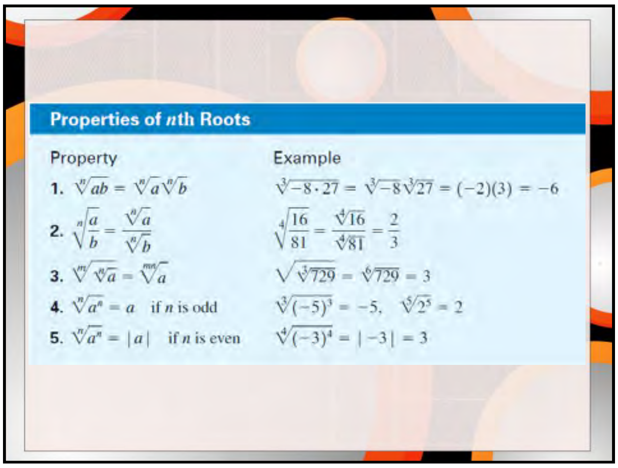
nth Root

Definition of nth Root

If n is any positive integer, then the **principal nth root** of a is defined as follows:

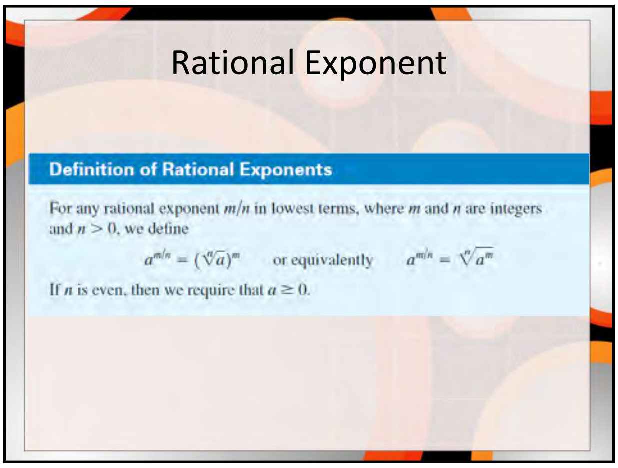
$$\sqrt[n]{a} = b \qquad \text{means} \qquad b^n = a$$

If n is even, we must have $a \geq 0$ and $b \geq 0$.



Properties of nth Roots

Property	Example
1. $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$	$\sqrt[3]{-8 \cdot 27} = \sqrt[3]{-8}\sqrt[3]{27} = (-2)(3) = -6$
2. $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$	$\sqrt[4]{\frac{16}{81}} = \frac{\sqrt[4]{16}}{\sqrt[4]{81}} = \frac{2}{3}$
3. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$	$\sqrt{\sqrt[3]{729}} = \sqrt[6]{729} = 3$
4. $\sqrt[n]{a^n} = a$ if n is odd	$\sqrt[3]{(-5)^3} = -5, \quad \sqrt[5]{2^5} = 2$
5. $\sqrt[n]{a^n} = \lvert a \rvert$ if n is even	$\sqrt[4]{(-3)^4} = \lvert -3 \rvert = 3$



Rational Exponent

Definition of Rational Exponents

For any rational exponent m/n in lowest terms, where m and n are integers and $n > 0$, we define

$$a^{m/n} = (\sqrt[n]{a})^m \qquad \text{or equivalently} \qquad a^{m/n} = \sqrt[n]{a^m}$$

If n is even, then we require that $a \geq 0$.

Exercises

15 – 17: Evaluate each expression.

15. $\sqrt[3]{\dfrac{8}{27}}$

16. $\sqrt[3]{\dfrac{-1}{64}}$

17. $\dfrac{\sqrt[5]{-3}}{\sqrt[5]{96}}$

18 – 21: Simplify the expression and eliminate any negative exponent(s). Assume that all letters denote positive numbers.

18. $\left(\dfrac{3a^{-2}}{4b^{-\frac{1}{3}}}\right)^{-1}$

19. $\dfrac{(y^{10}z^{-5})^{\frac{1}{5}}}{(y^{-2}z^{3})^{\frac{1}{3}}}$

20. $\dfrac{(9st)^{\frac{3}{2}}}{(27s^{3}t^{-4})^{\frac{2}{3}}}$

21. $\left(\dfrac{a^{2}b^{-3}}{x^{-1}y^{2}}\right)^{3}\left(\dfrac{x^{-2}b^{-1}}{a^{\frac{3}{2}}y^{\frac{1}{3}}}\right)$

Meeting 2

Function - 1

Definition of Function

DEFINITION A **function** f from a set D to a set Y is a rule that assigns a *unique* (single) element $f(x) \in Y$ to each element $x \in D$.

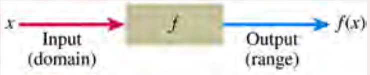
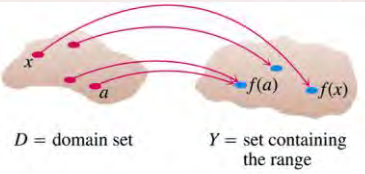


FIGURE 1.1 A diagram showing a function as a kind of machine.



A function from a set D to a set Y assigns a unique element of Y to each element in D.

Evaluating a Function

In the definition of a function the independent variable x plays the role of a "placeholder." For example, the function $f(x) = 3x^2 + x - 5$ can be thought of as

$$f(\square) = 3 \cdot \square^2 + \square - 5$$

To evaluate f at a number, we substitute the number for the placeholder.

Examples

Let $f(x) = 3x^2 + x - 5$. Evaluate each function value.

(a) $f(-2)$ (b) $f(0)$ (c) $f(4)$ (d) $f(\frac{1}{2})$

Solution To evaluate f at a number, we substitute the number for x in the definition of f.

(a) $f(-2) = 3 \cdot (-2)^2 + (-2) - 5 = 5$

(b) $f(0) = 3 \cdot 0^2 + 0 - 5 = -5$

(c) $f(4) = 3 \cdot 4^2 + 4 - 5 = 47$

(d) $f(\frac{1}{2}) = 3 \cdot (\frac{1}{2})^2 + \frac{1}{2} - 5 = -\frac{15}{4}$

Domain

Domain of a function is the set of all inputs for the function.

Examples

Find the domain of each function.

(a) $f(x) = \dfrac{1}{x^2 - x}$ (b) $g(x) = \sqrt{9 - x^2}$ (c) $h(t) = \dfrac{t}{\sqrt{t + 1}}$

Solution

(a) The function is not defined when the denominator is 0. Since

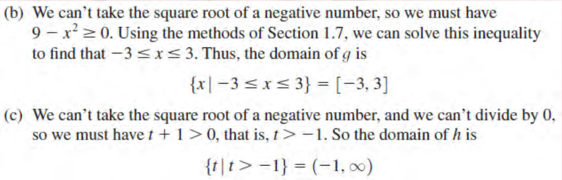
$$f(x) = \frac{1}{x^2 - x} = \frac{1}{x(x - 1)}$$

we see that $f(x)$ is not defined when $x = 0$ or $x = 1$. Thus, the domain of f is

$$\{x \mid x \neq 0, x \neq 1\}$$

The domain may also be written in interval notation as

$$(\infty, 0) \cup (0, 1) \cup (1, \infty)$$



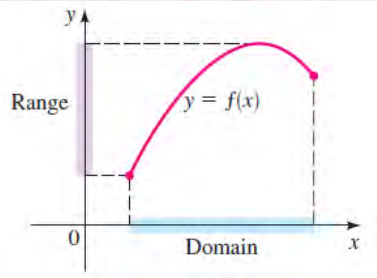
(b) We can't take the square root of a negative number, so we must have $9 - x^2 \geq 0$. Using the methods of Section 1.7, we can solve this inequality to find that $-3 \leq x \leq 3$. Thus, the domain of g is

$$\{x \mid -3 \leq x \leq 3\} = [-3, 3]$$

(c) We can't take the square root of a negative number, and we can't divide by 0, so we must have $t + 1 > 0$, that is, $t > -1$. So the domain of h is

$$\{t \mid t > -1\} = (-1, \infty)$$

The graph of a function helps us picture the domain and range of the function on the *x*-axis and *y*-axis



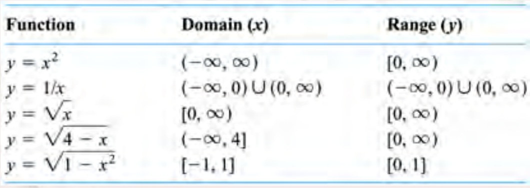
Exercises

1 – 2: Find $f(a)$, $f(a+h)$, and the difference quotient $\dfrac{f(a+h)-f(a)}{h}$, where $h \neq 0$.

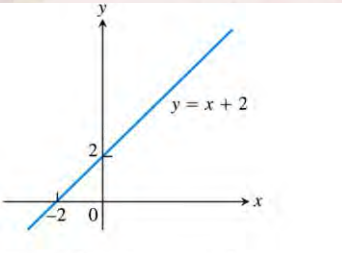
1. $f(x) = 3x + 2$
2. $f(x) = x^2 + 1$

3 – 4: Find the domain of the function.

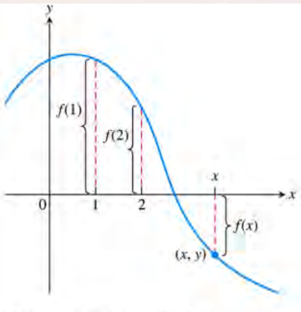
3. $f(x) = \dfrac{\sqrt{2+x}}{3-x}$
4. $f(x) = \dfrac{\sqrt{x}}{2x^2 + x - 1}$



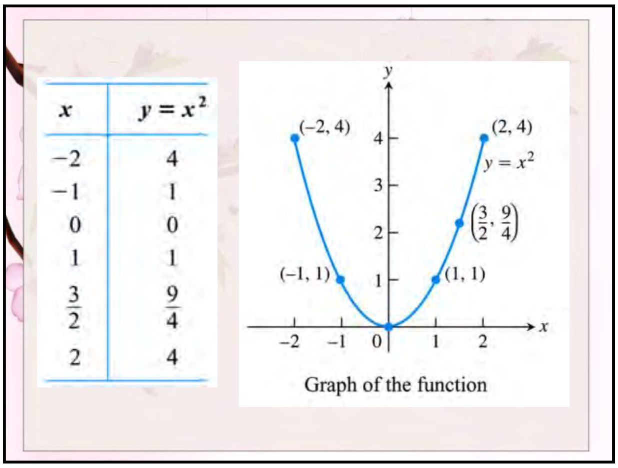
Function	**Domain (*x*)**	**Range (*y*)**
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$



The graph of $f(x) = x + 2$ is the set of points (x, y) for which y has the value $x + 2$.

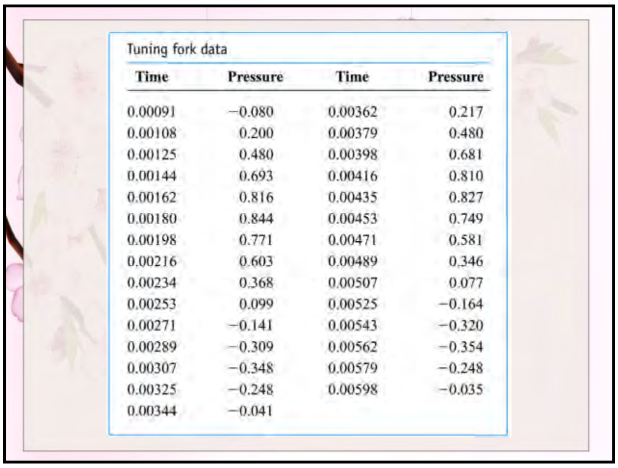


If (x, y) lies on the graph of f, then the value $y = f(x)$ is the height of the graph above the point x (or below x if $f(x)$ is negative).



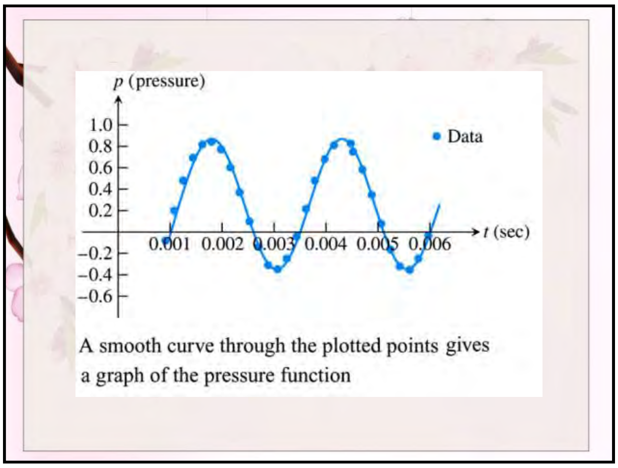
x	$y = x^2$
-2	4
-1	1
0	0
1	1
$\frac{3}{2}$	$\frac{9}{4}$
2	4

Graph of the function

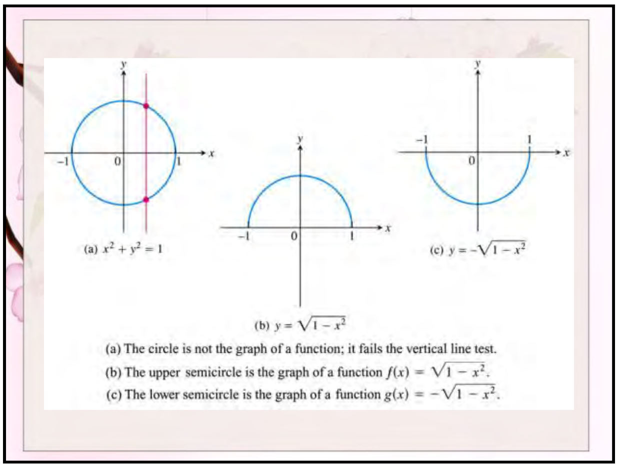


Tuning fork data

Time	Pressure	Time	Pressure
0.00091	−0.080	0.00362	0.217
0.00108	0.200	0.00379	0.480
0.00125	0.480	0.00398	0.681
0.00144	0.693	0.00416	0.810
0.00162	0.816	0.00435	0.827
0.00180	0.844	0.00453	0.749
0.00198	0.771	0.00471	0.581
0.00216	0.603	0.00489	0.346
0.00234	0.368	0.00507	0.077
0.00253	0.099	0.00525	−0.164
0.00271	−0.141	0.00543	−0.320
0.00289	−0.309	0.00562	−0.354
0.00307	−0.348	0.00579	−0.248
0.00325	−0.248	0.00598	−0.035
0.00344	−0.041		



A smooth curve through the plotted points gives a graph of the pressure function



(a) $x^2 + y^2 = 1$

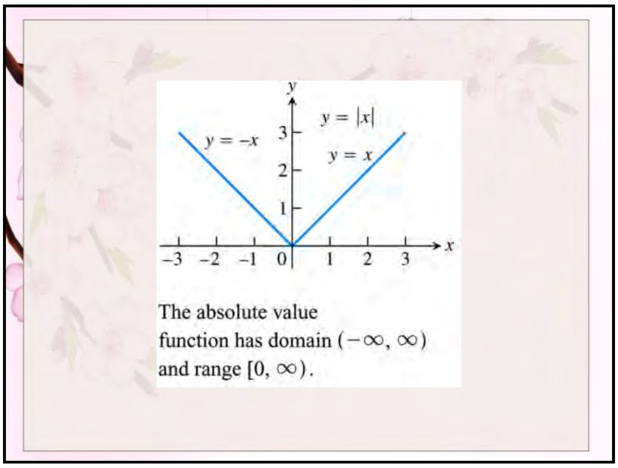
(b) $y = \sqrt{1 - x^2}$

(c) $y = -\sqrt{1 - x^2}$

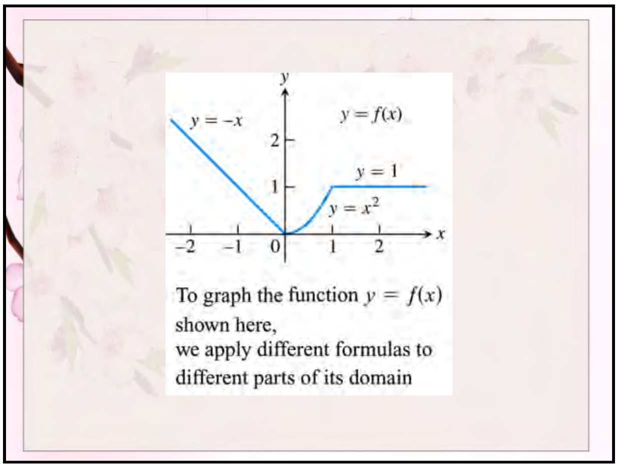
(a) The circle is not the graph of a function; it fails the vertical line test.

(b) The upper semicircle is the graph of a function $f(x) = \sqrt{1 - x^2}$.

(c) The lower semicircle is the graph of a function $g(x) = -\sqrt{1 - x^2}$.



The absolute value function has domain $(-\infty, \infty)$ and range $[0, \infty)$.



To graph the function $y = f(x)$ shown here, we apply different formulas to different parts of its domain

Polynomial Function

Functions defined by polynomial expressions are called polynomial functions, for example

$$P(x) = 2x^3 - x + 1$$

Polynomial functions are easy to evaluate because they are defined using only addition, subtraction, and multiplication

Polynomial Functions

A **polynomial function of degree n** is a function of the form

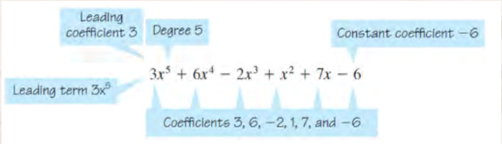
$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where n is a nonnegative integer and $a_n \neq 0$.

The numbers $a_0, a_1, a_2, \ldots, a_n$ are called the **coefficients** of the polynomial.

The number a_0 is the **constant coefficient** or **constant term**.

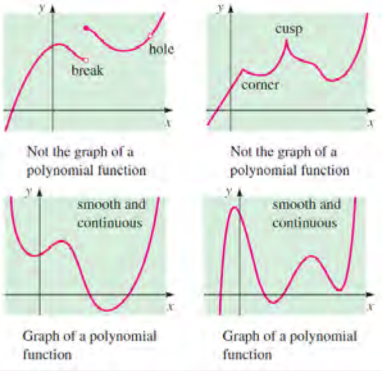
The number a_n, the coefficient of the highest power, is the **leading coefficient**, and the term $a_n x^n$ is the **leading term**.



$P(x) = 3$	Degree 0
$Q(x) = 4x - 7$	Degree 1
$R(x) = x^2 + x$	Degree 2
$S(x) = 2x^3 - 6x^2 - 10$	Degree 3

If a polynomial consists of just a single term, then it is called a **monomial**.

The graph of a polynomial function is always a smooth curve; that is, it has no breaks or corners.



Not the graph of a polynomial function

Not the graph of a polynomial function

Graph of a polynomial function

Graph of a polynomial function

Quadratic Functions

A **quadratic function** is a function f of the form

$$f(x) = ax^2 + bx + c$$

Maximum or Minimum Value of a Quadratic Function

The maximum or minimum value of a quadratic function $f(x) = ax^2 + bx + c$ occurs at

$$x = -\frac{b}{2a}$$

If $a > 0$, then the **minimum value** is $f\left(-\frac{b}{2a}\right)$.

If $a < 0$, then the **maximum value** is $f\left(-\frac{b}{2a}\right)$.

Examples

Find the maximum or minimum value of each quadratic function.

(a) $f(x) = x^2 + 4x$ (b) $g(x) = -2x^2 + 4x - 5$

Solution

(a) This is a quadratic function with $a = 1$ and $b = 4$. Thus, the maximum or minimum value occurs at

$$x = -\frac{b}{2a} = -\frac{4}{2 \cdot 1} = -2$$

Since $a > 0$, the function has the *minimum* value

$$f(-2) = (-2)^2 + 4(-2) = -4$$

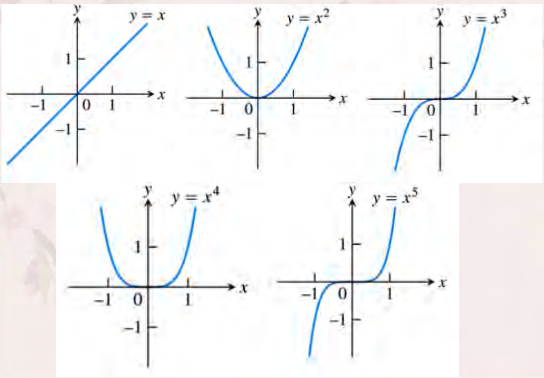
(b) This is a quadratic function with $a = -2$ and $b = 4$. Thus, the maximum or minimum value occurs at

$$x = -\frac{b}{2a} = -\frac{4}{2\cdot(-2)} = 1$$

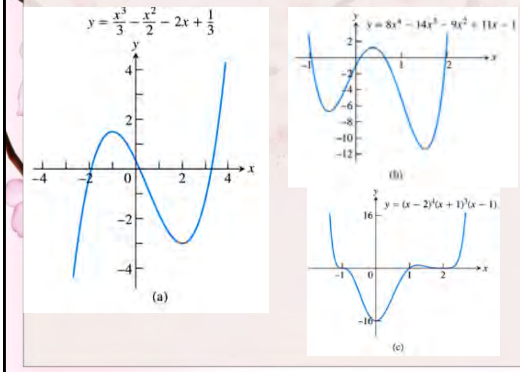
Since $a < 0$, the function has the *maximum* value

$$f(1) = -2(1)^2 + 4(1) - 5 = -3$$

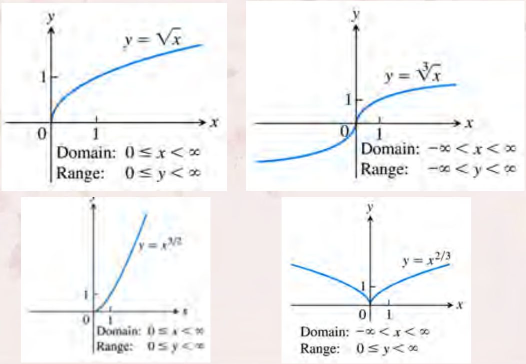
Graphs of $f(x) = x^n$, $n = 1, 2, 3, 4, 5$, defined for $-\infty < x < \infty$



Graphs of three polynomial functions.



Graphs of the power functions $f(x) = x^a$ for $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}$, and $\frac{2}{3}$



Rational Function

A rational function is a function of the form

$$r(x) = \frac{P(x)}{Q(x)}$$

The *domain* of a rational function consists of all real numbers x except those for which the denominator is zero.

Example

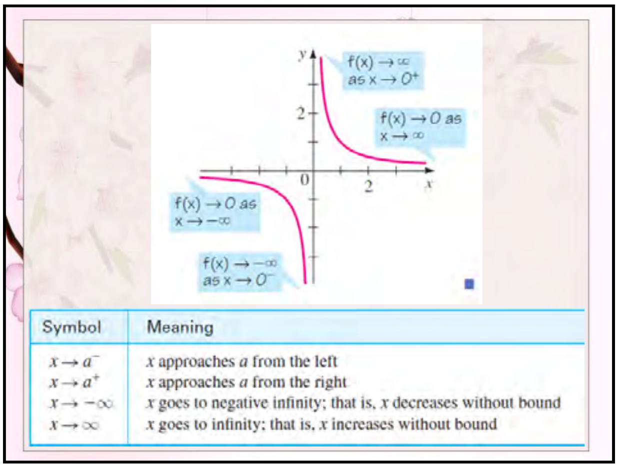
Sketch a graph of the rational function $f(x) = \frac{1}{x}$.

x	$f(x)$
−0.1	−10
−0.01	−100
−0.00001	−100,000

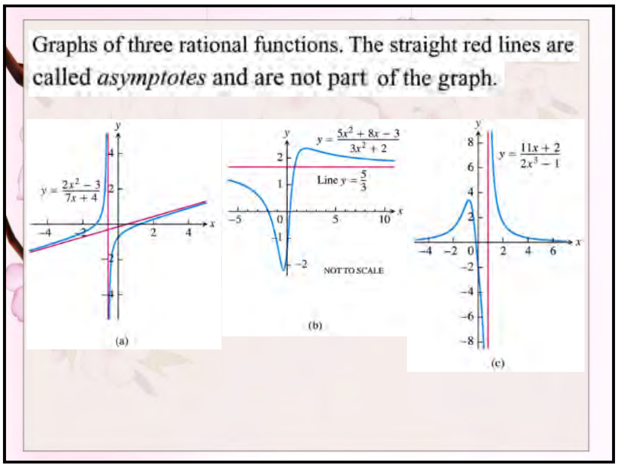
Approaching 0^- Approaching $-\infty$

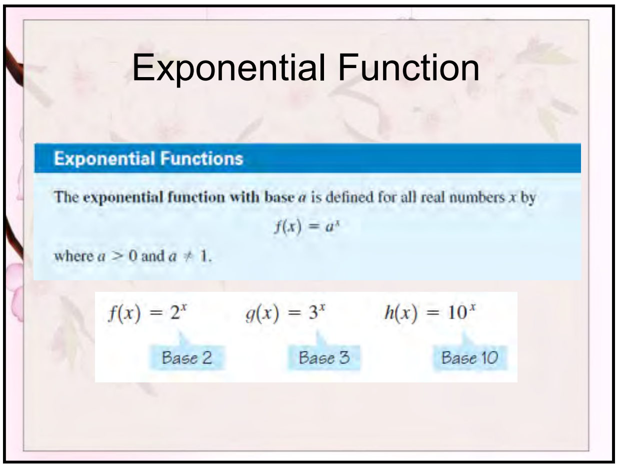
x	$f(x)$
0.1	10
0.01	100
0.00001	100,000

Approaching 0^+ Approaching ∞



Symbol	Meaning
$x \to a^-$	x approaches a from the left
$x \to a^+$	x approaches a from the right
$x \to -\infty$	x goes to negative infinity; that is, x decreases without bound
$x \to \infty$	x goes to infinity; that is, x increases without bound


Graphs of three rational functions. The straight red lines are called *asymptotes* and are not part of the graph.


Exponential Function

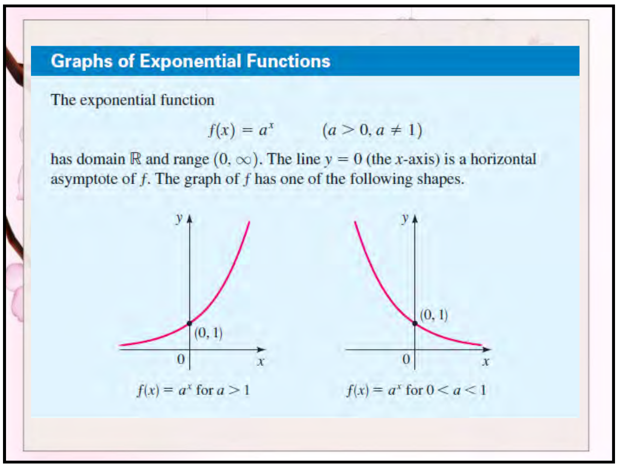
Exponential Functions

The exponential function with base a is defined for all real numbers x by

$$f(x) = a^x$$

where $a > 0$ and $a \neq 1$.

$f(x) = 2^x$ $\quad g(x) = 3^x$ $\quad h(x) = 10^x$

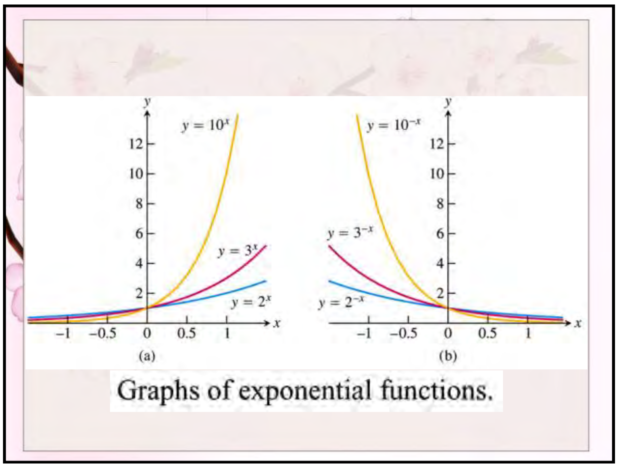

Graphs of Exponential Functions

The exponential function

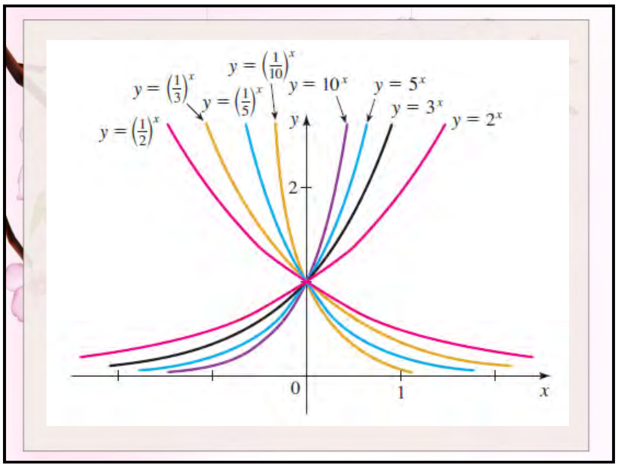
$$f(x) = a^x \qquad (a > 0, a \neq 1)$$

has domain ℝ and range $(0, \infty)$. The line $y = 0$ (the x-axis) is a horizontal asymptote of f. The graph of f has one of the following shapes.

$f(x) = a^x$ for $a > 1$ $\qquad$ $f(x) = a^x$ for $0 < a < 1$



Graphs of exponential functions.

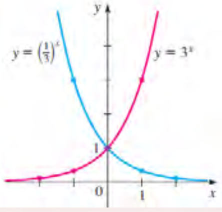


Example

Draw the graph of each function.

(a) $f(x) = 3^x$ (b) $g(x) = \left(\frac{1}{3}\right)^x$

x	$f(x) = 3^x$	$g(x) = \left(\frac{1}{3}\right)^x$
-3	$\frac{1}{27}$	27
-2	$\frac{1}{9}$	9
-1	$\frac{1}{3}$	3
0	1	1
1	3	$\frac{1}{3}$
2	9	$\frac{1}{9}$
3	27	$\frac{1}{27}$



The Natural Exponential Function

The **natural exponential function** is the exponential function

$$f(x) = e^x$$

with base e. It is often referred to as *the* exponential function.

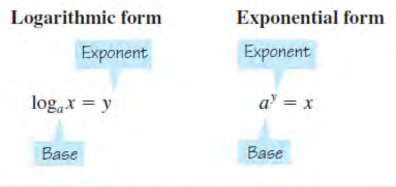
Logarithmic Function

Definition of the Logarithmic Function

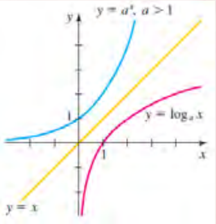
Let a be a positive number with $a \neq 1$. The **logarithmic function with base** a, denoted by $\log_a$, is defined by

$$\log_a x = y \quad \Leftrightarrow \quad a^y = x$$

So, $\log_a x$ is the *exponent* to which the base a must be raised to give x.



Logarithmic form	Exponential form
$\log_{10} 100{,}000 = 5$	$10^5 = 100{,}000$
$\log_2 8 = 3$	$2^3 = 8$
$\log_2\left(\frac{1}{8}\right) = -3$	$2^{-3} = \frac{1}{8}$
$\log_5 s = r$	$5^r = s$



The function $f(x) = \log_a x$ has domain $(0, \infty)$ and range $\mathbb{R}$.

Properties of Logarithms

Property	Reason
1. $\log_a 1 = 0$	We must raise a to the power 0 to get 1.
2. $\log_a a = 1$	We must raise a to the power 1 to get a.
3. $\log_a a^x = x$	We must raise a to the power x to get a^x.
4. $a^{\log_a x} = x$	$\log_a x$ is the power to which a must be raised to get x.

Common Logarithm

The logarithm with base 10 is called the **common logarithm** and is denoted by omitting the base:

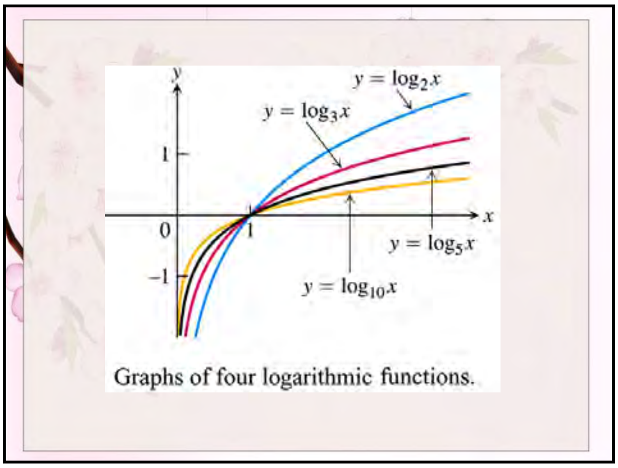
$$\log x = \log_{10} x$$

Natural Logarithm

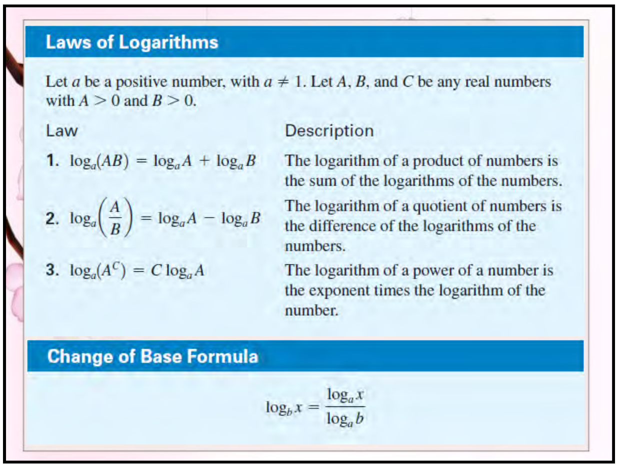
The logarithm with base e is called the **natural logarithm** and is denoted by **ln**:

$$\ln x = \log_e x$$

$$\ln x = y \quad \Leftrightarrow \quad e^y = x$$



Graphs of four logarithmic functions.



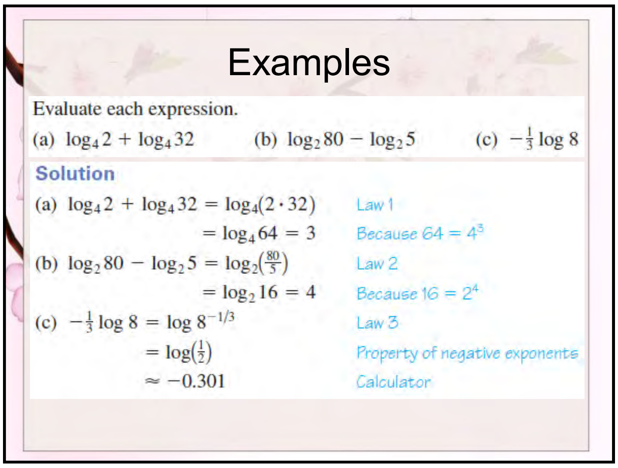
Laws of Logarithms

Let a be a positive number, with $a \neq 1$. Let A, B, and C be any real numbers with $A > 0$ and $B > 0$.

Law	Description
1. $\log_a(AB) = \log_a A + \log_a B$	The logarithm of a product of numbers is the sum of the logarithms of the numbers.
2. $\log_a\left(\frac{A}{B}\right) = \log_a A - \log_a B$	The logarithm of a quotient of numbers is the difference of the logarithms of the numbers.
3. $\log_a(A^C) = C \log_a A$	The logarithm of a power of a number is the exponent times the logarithm of the number.

Change of Base Formula

$$\log_b x = \frac{\log_a x}{\log_a b}$$



Examples

Evaluate each expression.

(a) $\log_4 2 + \log_4 32$ (b) $\log_2 80 - \log_2 5$ (c) $-\frac{1}{3}\log 8$

Solution

(a) $\log_4 2 + \log_4 32 = \log_4(2 \cdot 32)$ Law 1

$= \log_4 64 = 3$ Because $64 = 4^3$

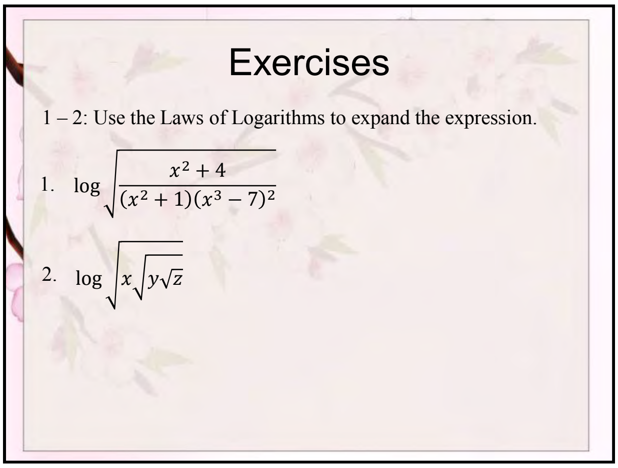
(b) $\log_2 80 - \log_2 5 = \log_2(\frac{80}{5})$ Law 2

$= \log_2 16 = 4$ Because $16 = 2^4$

(c) $-\frac{1}{3}\log 8 = \log 8^{-1/3}$ Law 3

$= \log(\frac{1}{2})$ Property of negative exponents

≈ -0.301 Calculator

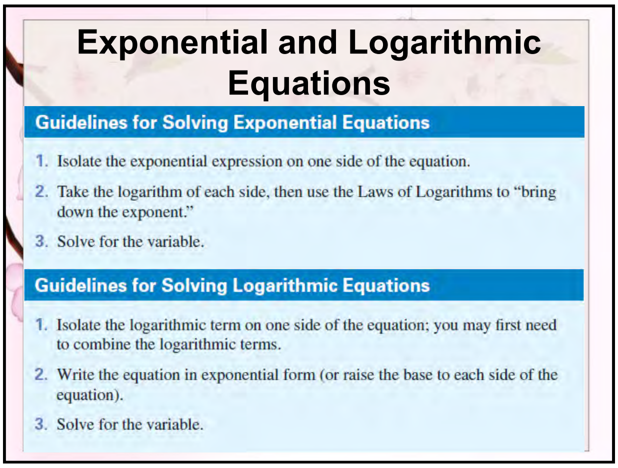


Exercises

1 – 2: Use the Laws of Logarithms to expand the expression.

1. $\log \sqrt{\dfrac{x^2+4}{(x^2+1)(x^3-7)^2}}$

2. $\log \sqrt{x\sqrt{y\sqrt{z}}}$



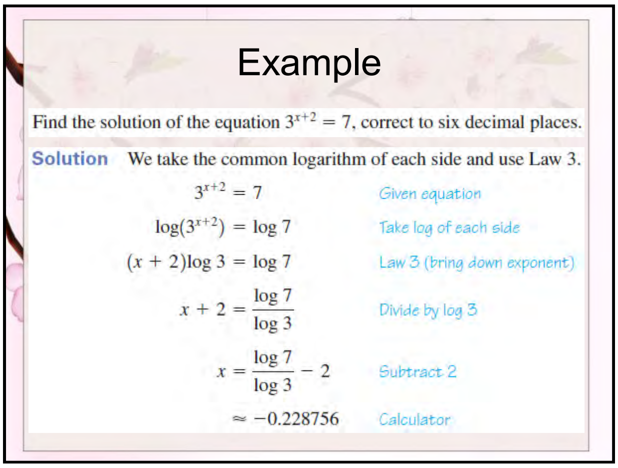
Exponential and Logarithmic Equations

Guidelines for Solving Exponential Equations

1. Isolate the exponential expression on one side of the equation.
2. Take the logarithm of each side, then use the Laws of Logarithms to "bring down the exponent."
3. Solve for the variable.

Guidelines for Solving Logarithmic Equations

1. Isolate the logarithmic term on one side of the equation; you may first need to combine the logarithmic terms.
2. Write the equation in exponential form (or raise the base to each side of the equation).
3. Solve for the variable.



Example

Find the solution of the equation $3^{x+2} = 7$, correct to six decimal places.

Solution We take the common logarithm of each side and use Law 3.

$3^{x+2} = 7$ Given equation

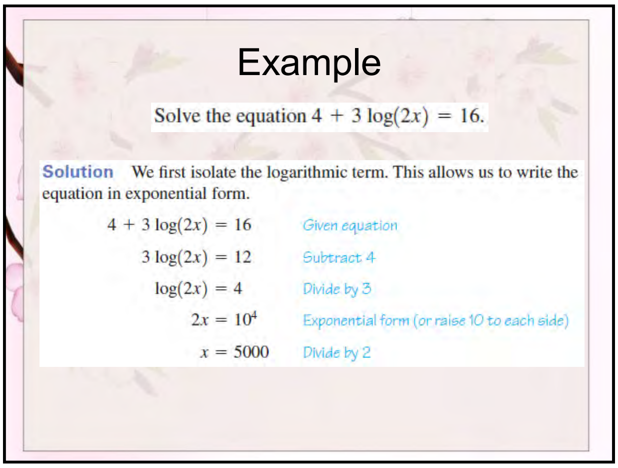
$\log(3^{x+2}) = \log 7$ Take log of each side

$(x + 2)\log 3 = \log 7$ Law 3 (bring down exponent)

$x + 2 = \dfrac{\log 7}{\log 3}$ Divide by log 3

$x = \dfrac{\log 7}{\log 3} - 2$ Subtract 2

≈ -0.228756 Calculator



Example

Solve the equation $4 + 3\log(2x) = 16$.

Solution We first isolate the logarithmic term. This allows us to write the equation in exponential form.

$4 + 3\log(2x) = 16$ Given equation

$3\log(2x) = 12$ Subtract 4

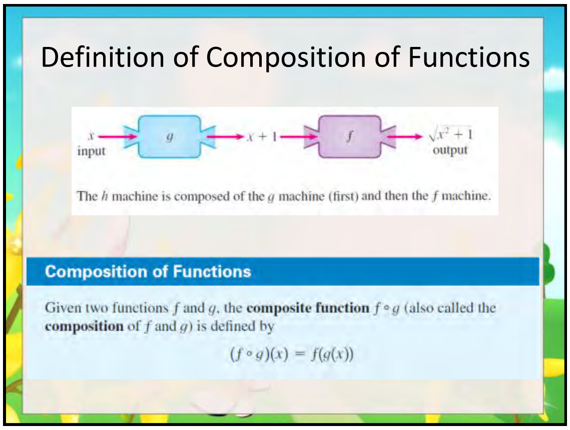
$\log(2x) = 4$ Divide by 3

$2x = 10^4$ Exponential form (or raise 10 to each side)

$x = 5000$ Divide by 2

Meeting 3

Function - 2



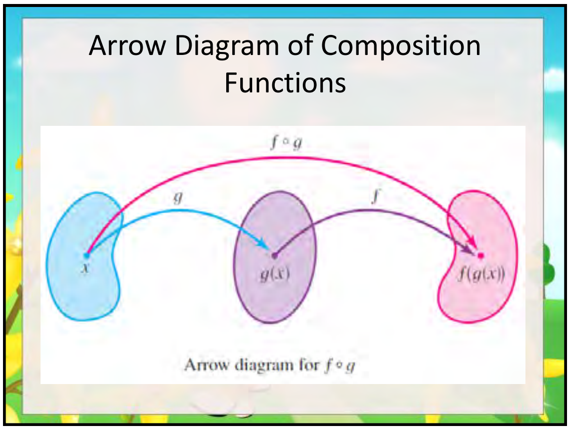
Definition of Composition of Functions

The h machine is composed of the g machine (first) and then the f machine.

Composition of Functions

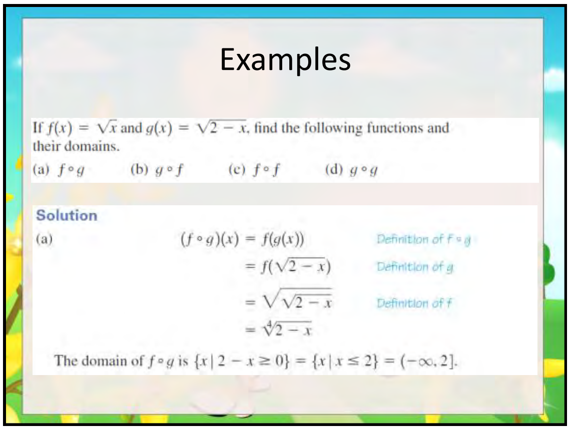
Given two functions f and g, the **composite function** $f \circ g$ (also called the **composition** of f and g) is defined by

$$(f \circ g)(x) = f(g(x))$$



Arrow Diagram of Composition Functions

Arrow diagram for $f \circ g$



Examples

If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{2 - x}$, find the following functions and their domains.

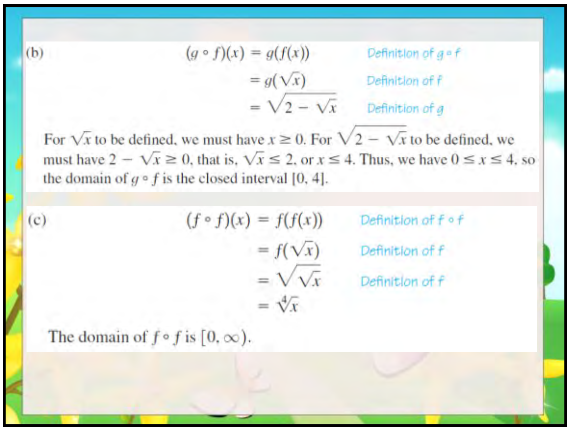
(a) $f \circ g$ (b) $g \circ f$ (c) $f \circ f$ (d) $g \circ g$

Solution

(a)

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) && \text{Definition of } f \circ g\\ &= f(\sqrt{2 - x}) && \text{Definition of } g\\ &= \sqrt{\sqrt{2 - x}} && \text{Definition of } f\\ &= \sqrt[4]{2 - x}\end{aligned}$$

The domain of $f \circ g$ is $\{x \mid 2 - x \geq 0\} = \{x \mid x \leq 2\} = (-\infty, 2]$.



(b)

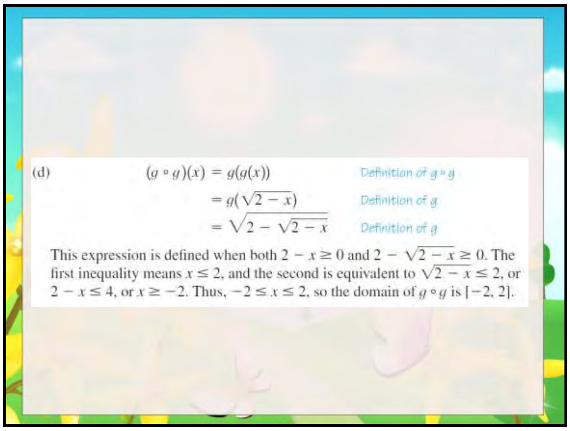
$$\begin{aligned}(g \circ f)(x) &= g(f(x)) && \text{Definition of } g \circ f\\ &= g(\sqrt{x}) && \text{Definition of } f\\ &= \sqrt{2 - \sqrt{x}} && \text{Definition of } g\end{aligned}$$

For $\sqrt{x}$ to be defined, we must have $x \geq 0$. For $\sqrt{2 - \sqrt{x}}$ to be defined, we must have $2 - \sqrt{x} \geq 0$, that is, $\sqrt{x} \leq 2$, or $x \leq 4$. Thus, we have $0 \leq x \leq 4$, so the domain of $g \circ f$ is the closed interval $[0, 4]$.

(c)

$$\begin{aligned}(f \circ f)(x) &= f(f(x)) && \text{Definition of } f \circ f\\ &= f(\sqrt{x}) && \text{Definition of } f\\ &= \sqrt{\sqrt{x}} && \text{Definition of } f\\ &= \sqrt[4]{x}\end{aligned}$$

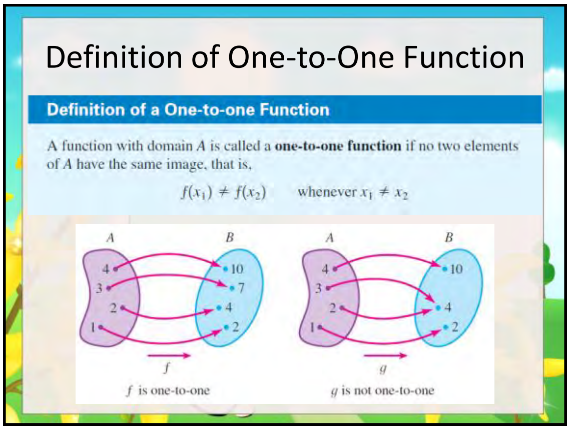
The domain of $f \circ f$ is $[0, \infty)$.



(d)

$$\begin{aligned}(g \circ g)(x) &= g(g(x)) && \text{Definition of } g \circ g\\ &= g(\sqrt{2 - x}) && \text{Definition of } g\\ &= \sqrt{2 - \sqrt{2 - x}} && \text{Definition of } g\end{aligned}$$

This expression is defined when both $2 - x \geq 0$ and $2 - \sqrt{2 - x} \geq 0$. The first inequality means $x \leq 2$, and the second is equivalent to $\sqrt{2 - x} \leq 2$, or $2 - x \leq 4$, or $x \geq -2$. Thus, $-2 \leq x \leq 2$, so the domain of $g \circ g$ is $[-2, 2]$.



Definition of One-to-One Function

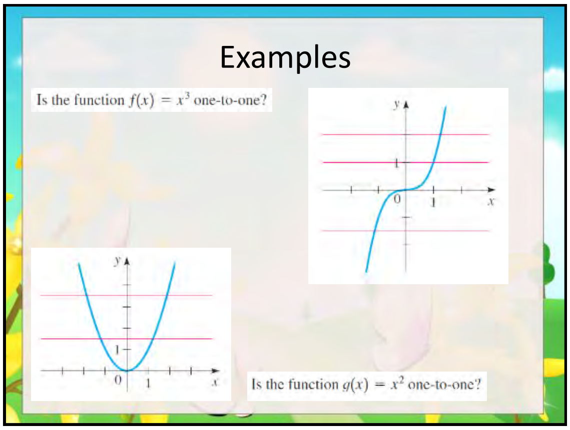
Definition of a One-to-one Function

A function with domain A is called a **one-to-one function** if no two elements of A have the same image, that is,

$$f(x_1) \neq f(x_2) \qquad \text{whenever } x_1 \neq x_2$$

f is one-to-one

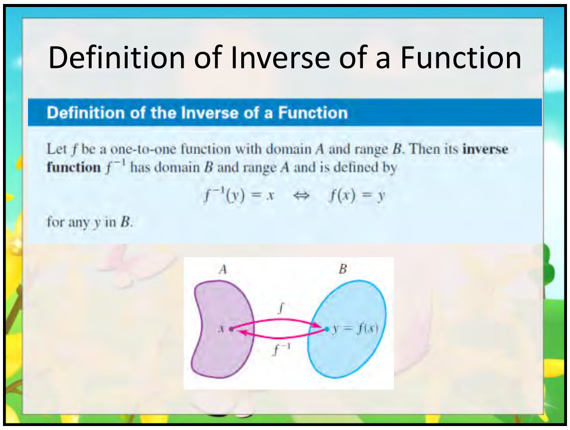
g is not one-to-one



Examples

Is the function $f(x) = x^3$ one-to-one?

Is the function $g(x) = x^2$ one-to-one?



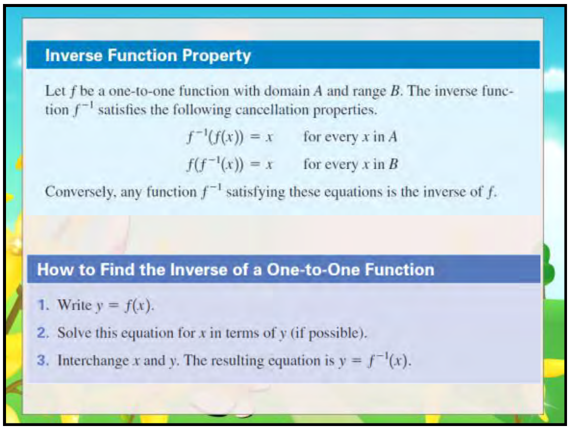
Definition of Inverse of a Function

Definition of the Inverse of a Function

Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y$$

for any y in B.



Inverse Function Property

Let f be a one-to-one function with domain A and range B. The inverse function f^{-1} satisfies the following cancellation properties.

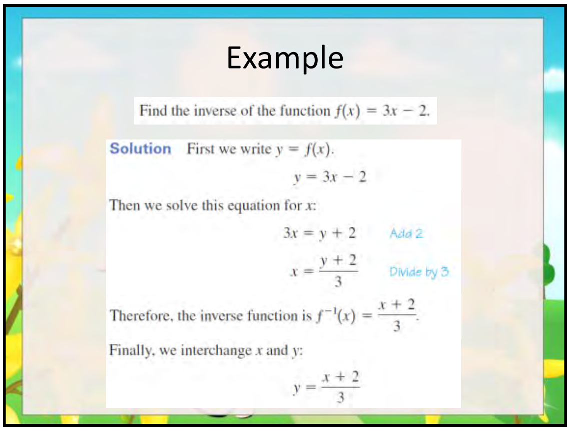
$$f^{-1}(f(x)) = x \qquad \text{for every } x \text{ in } A$$

$$f(f^{-1}(x)) = x \qquad \text{for every } x \text{ in } B$$

Conversely, any function f^{-1} satisfying these equations is the inverse of f.

How to Find the Inverse of a One-to-One Function

1. Write $y = f(x)$.
2. Solve this equation for x in terms of y (if possible).
3. Interchange x and y. The resulting equation is $y = f^{-1}(x)$.



Example

Find the inverse of the function $f(x) = 3x - 2$.

Solution First we write $y = f(x)$.

$$y = 3x - 2$$

Then we solve this equation for x:

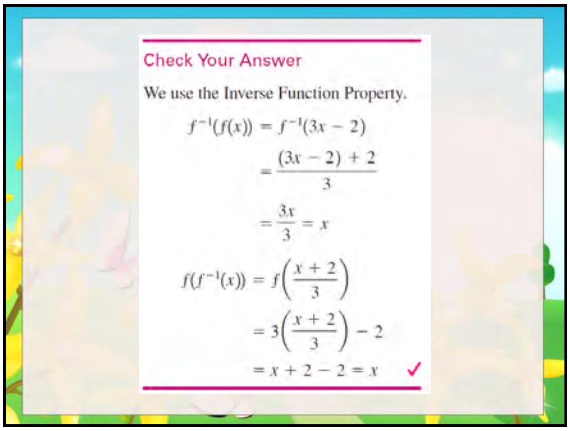
$$3x = y + 2 \qquad \text{Add 2}$$

$$x = \frac{y+2}{3} \qquad \text{Divide by 3}$$

Therefore, the inverse function is $f^{-1}(x) = \dfrac{x+2}{3}$.

Finally, we interchange x and y:

$$y = \frac{x+2}{3}$$



Check Your Answer

We use the Inverse Function Property.

$$f^{-1}(f(x)) = f^{-1}(3x-2)$$

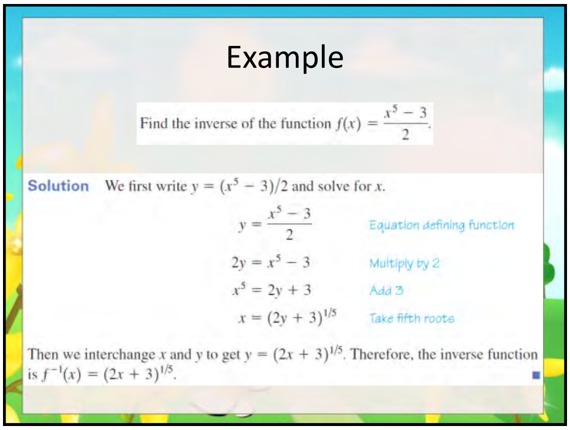
$$= \frac{(3x-2)+2}{3}$$

$$= \frac{3x}{3} = x$$

$$f(f^{-1}(x)) = f\left(\frac{x+2}{3}\right)$$

$$= 3\left(\frac{x+2}{3}\right) - 2$$

$$= x + 2 - 2 = x \quad ✓$$



Example

Find the inverse of the function $f(x) = \frac{x^5 - 3}{2}$.

Solution We first write $y = (x^5 - 3)/2$ and solve for x.

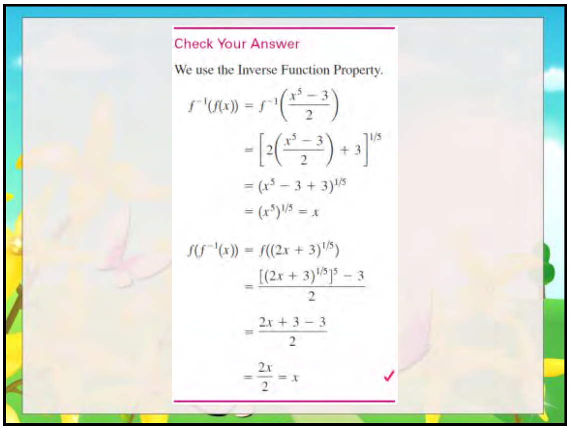
$$y = \frac{x^5 - 3}{2} \qquad \text{Equation defining function}$$

$$2y = x^5 - 3 \qquad \text{Multiply by 2}$$

$$x^5 = 2y + 3 \qquad \text{Add 3}$$

$$x = (2y + 3)^{1/5} \qquad \text{Take fifth roots}$$

Then we interchange x and y to get $y = (2x + 3)^{1/5}$. Therefore, the inverse function is $f^{-1}(x) = (2x + 3)^{1/5}$. ■

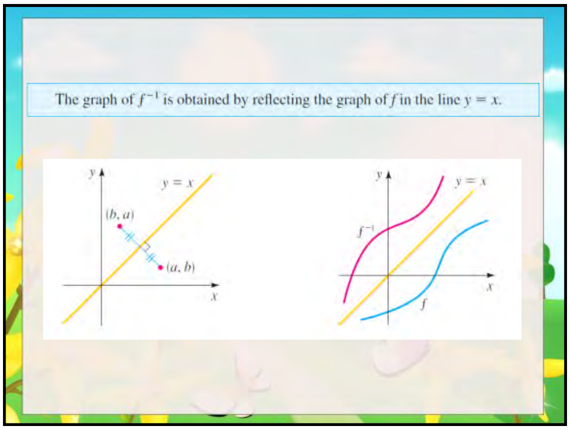


Check Your Answer

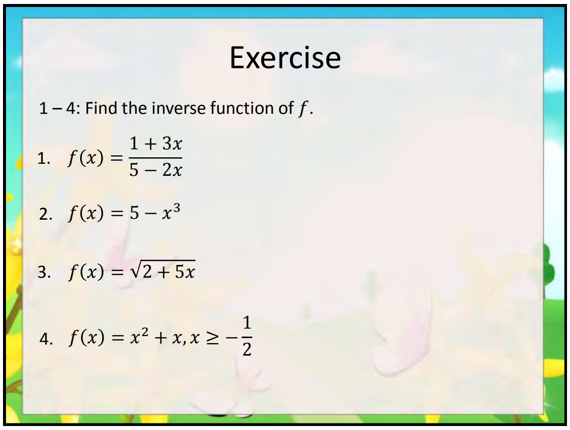
We use the Inverse Function Property.

$$f^{-1}(f(x)) = f^{-1}\left(\frac{x^5 - 3}{2}\right) = \left[2\left(\frac{x^5 - 3}{2}\right) + 3\right]^{1/5} = (x^5 - 3 + 3)^{1/5} = (x^5)^{1/5} = x$$

$$f(f^{-1}(x)) = f((2x + 3)^{1/5}) = \frac{[(2x + 3)^{1/5}]^5 - 3}{2} = \frac{2x + 3 - 3}{2} = \frac{2x}{2} = x \quad ✓$$



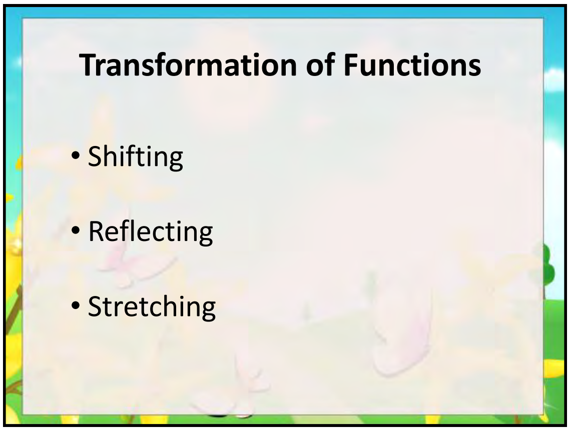
The graph of f^{-1} is obtained by reflecting the graph of f in the line $y = x$.



Exercise

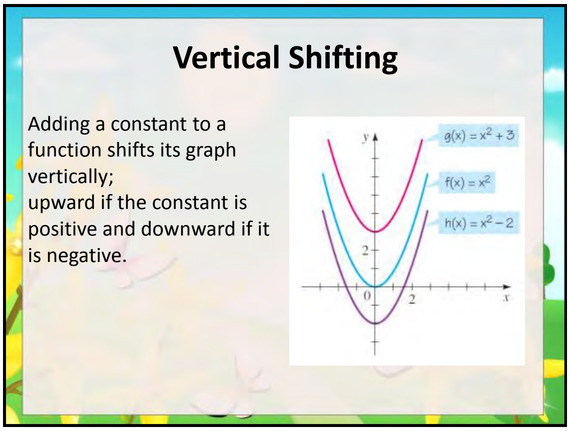
1 – 4: Find the inverse function of f.

1. $f(x) = \frac{1 + 3x}{5 - 2x}$
2. $f(x) = 5 - x^3$
3. $f(x) = \sqrt{2 + 5x}$
4. $f(x) = x^2 + x, x \geq -\frac{1}{2}$



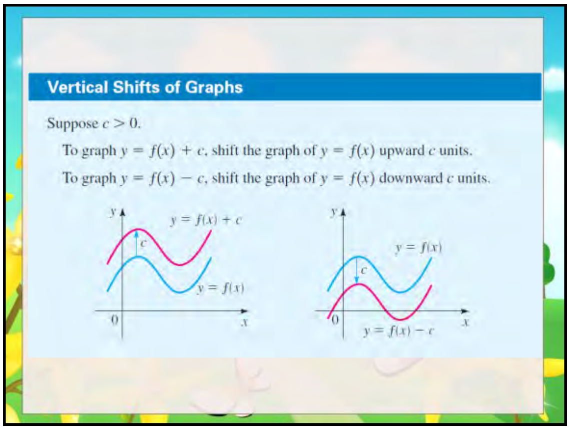
Transformation of Functions

- Shifting
- Reflecting
- Stretching



Vertical Shifting

Adding a constant to a function shifts its graph vertically; upward if the constant is positive and downward if it is negative.

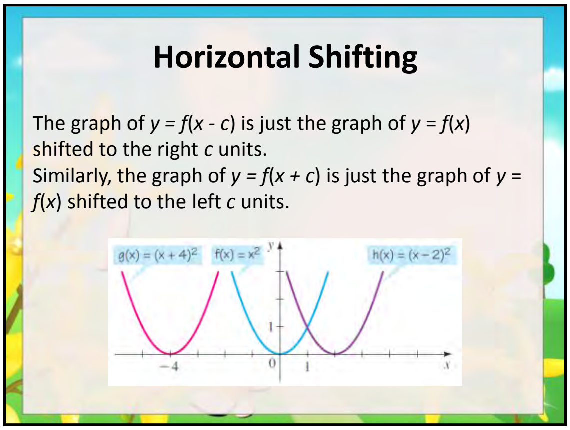


Vertical Shifts of Graphs

Suppose $c > 0$.

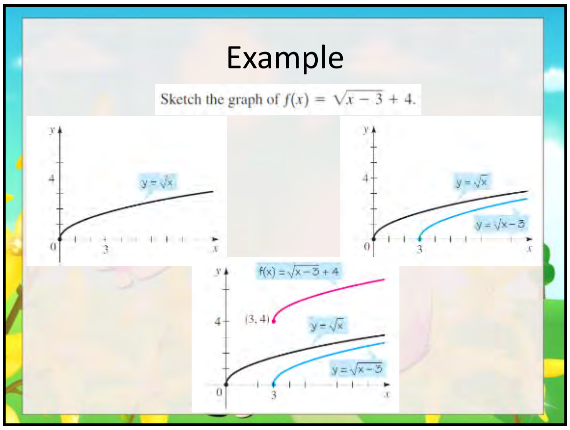
To graph $y = f(x) + c$, shift the graph of $y = f(x)$ upward c units.

To graph $y = f(x) - c$, shift the graph of $y = f(x)$ downward c units.



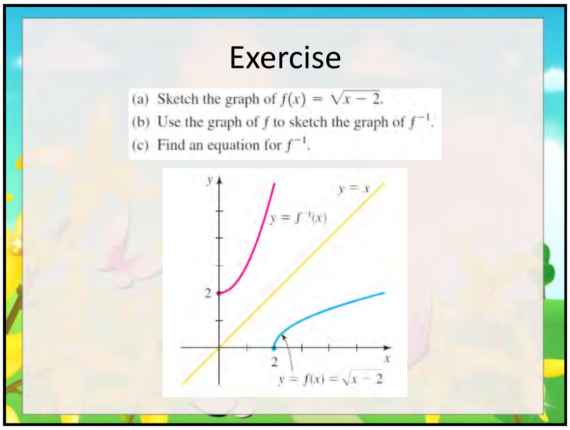
Horizontal Shifting

The graph of *y* = *f*(*x* - *c*) is just the graph of *y* = *f*(*x*) shifted to the right *c* units.
Similarly, the graph of *y* = *f*(*x* + *c*) is just the graph of *y* = *f*(*x*) shifted to the left *c* units.



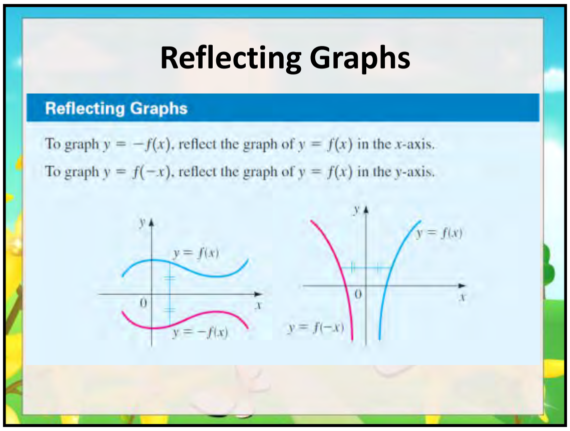
Example

Sketch the graph of $f(x) = \sqrt{x - 3} + 4$.



Exercise

(a) Sketch the graph of $f(x) = \sqrt{x - 2}$.
(b) Use the graph of f to sketch the graph of f^{-1}.
(c) Find an equation for f^{-1}.

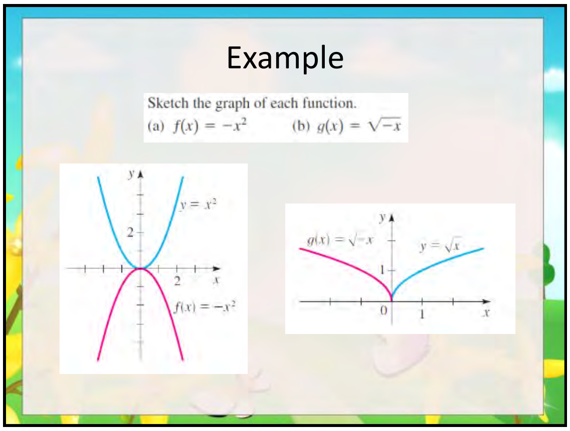


Reflecting Graphs

Reflecting Graphs

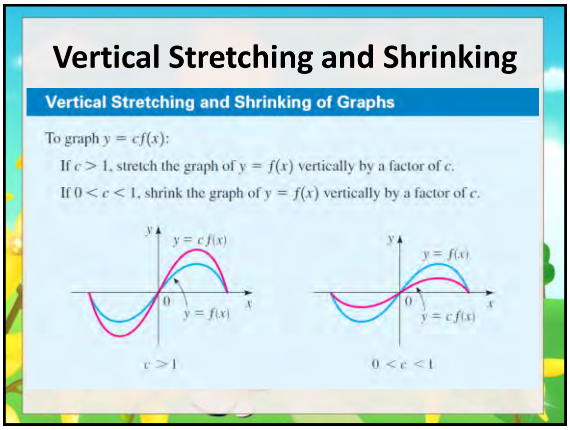
To graph $y = -f(x)$, reflect the graph of $y = f(x)$ in the x-axis.

To graph $y = f(-x)$, reflect the graph of $y = f(x)$ in the y-axis.



Example

Sketch the graph of each function.
(a) $f(x) = -x^2$ (b) $g(x) = \sqrt{-x}$



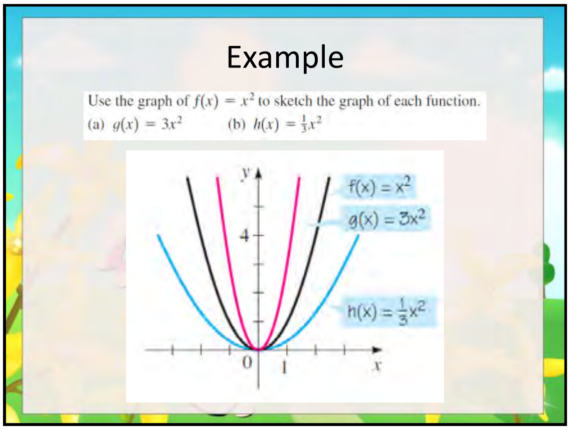
Vertical Stretching and Shrinking

Vertical Stretching and Shrinking of Graphs

To graph $y = cf(x)$:

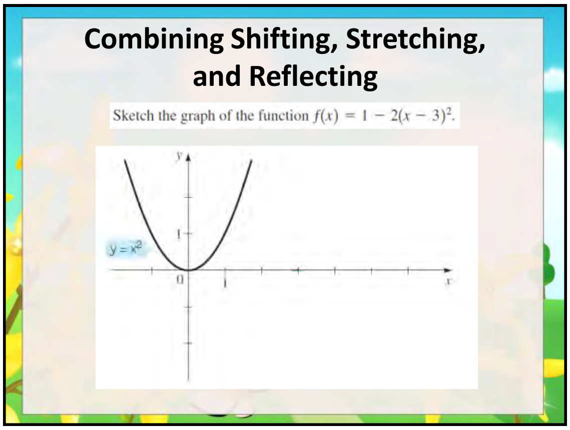
If $c > 1$, stretch the graph of $y = f(x)$ vertically by a factor of c.

If $0 < c < 1$, shrink the graph of $y = f(x)$ vertically by a factor of c.



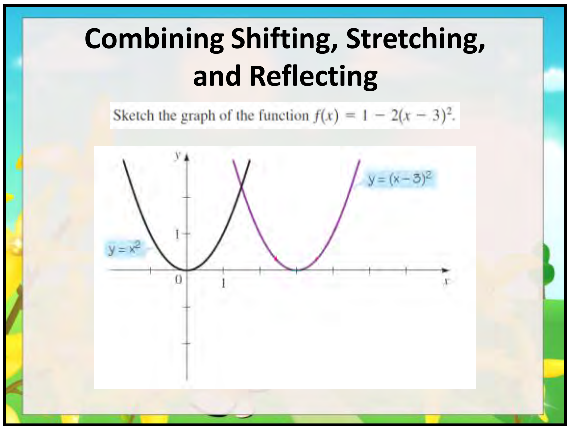
Example

Use the graph of $f(x) = x^2$ to sketch the graph of each function.
(a) $g(x) = 3x^2$ (b) $h(x) = \frac{1}{3}x^2$



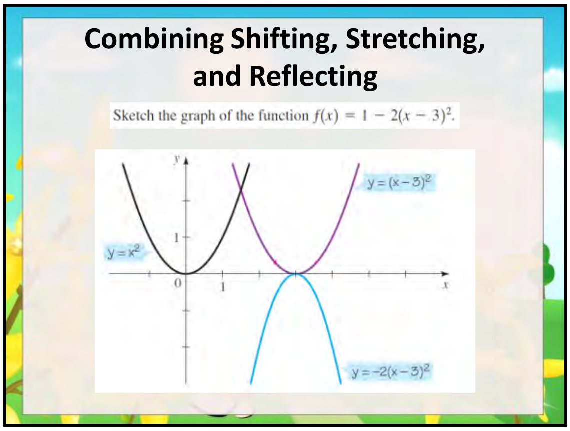
Combining Shifting, Stretching, and Reflecting

Sketch the graph of the function $f(x) = 1 - 2(x - 3)^2$.



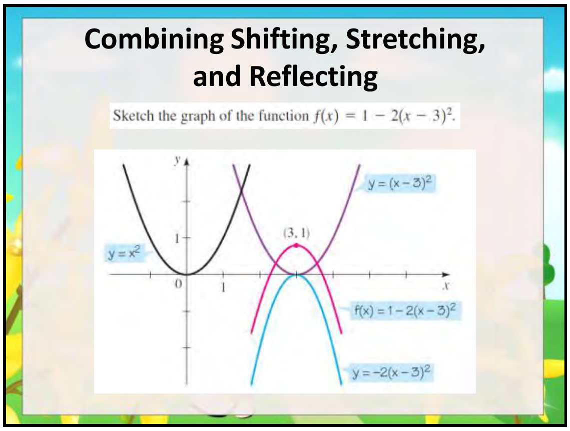
Combining Shifting, Stretching, and Reflecting

Sketch the graph of the function $f(x) = 1 - 2(x - 3)^2$.



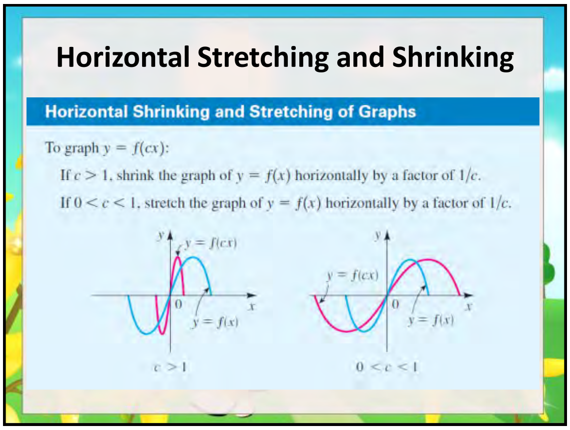
Combining Shifting, Stretching, and Reflecting

Sketch the graph of the function $f(x) = 1 - 2(x - 3)^2$.



Combining Shifting, Stretching, and Reflecting

Sketch the graph of the function $f(x) = 1 - 2(x - 3)^2$.



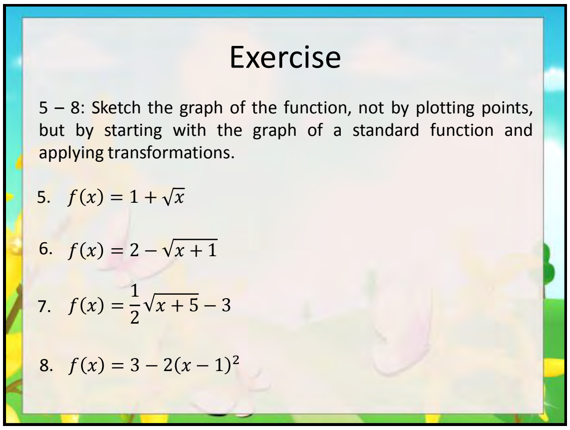
Horizontal Stretching and Shrinking

Horizontal Shrinking and Stretching of Graphs

To graph $y = f(cx)$:

If $c > 1$, shrink the graph of $y = f(x)$ horizontally by a factor of $1/c$.

If $0 < c < 1$, stretch the graph of $y = f(x)$ horizontally by a factor of $1/c$.



Exercise

5 – 8: Sketch the graph of the function, not by plotting points, but by starting with the graph of a standard function and applying transformations.

5. $f(x) = 1 + \sqrt{x}$
6. $f(x) = 2 - \sqrt{x+1}$
7. $f(x) = \frac{1}{2}\sqrt{x+5} - 3$
8. $f(x) = 3 - 2(x-1)^2$

Meeting 4

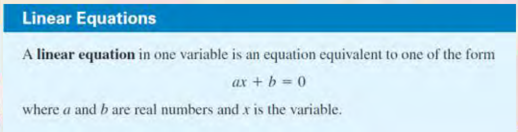
Equality and Inequality

Equations

An equation is a statement that two mathematical expressions are equal.

The values of the unknown that make the equation true are called the **solutions** or **roots** of the equation, and the process of finding the solutions is called **solving the equation**.

Linear Equations



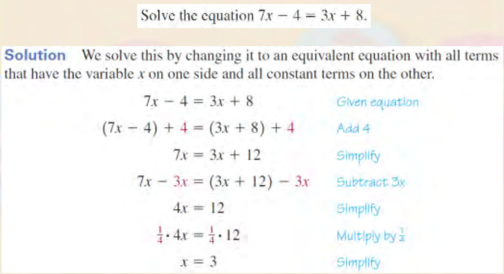
Linear Equations

A **linear equation** in one variable is an equation equivalent to one of the form

$$ax + b = 0$$

where a and b are real numbers and x is the variable.

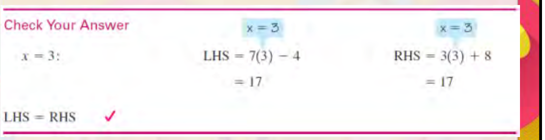
Example



Solve the equation $7x - 4 = 3x + 8$.

Solution We solve this by changing it to an equivalent equation with all terms that have the variable x on one side and all constant terms on the other.

$7x - 4 = 3x + 8$	Given equation
$(7x - 4) + 4 = (3x + 8) + 4$	Add 4
$7x = 3x + 12$	Simplify
$7x - 3x = (3x + 12) - 3x$	Subtract 3x
$4x = 12$	Simplify
$\frac{1}{4} \cdot 4x = \frac{1}{4} \cdot 12$	Multiply by $\frac{1}{4}$
$x = 3$	Simplify



Check Your Answer

$x = 3$:

$x = 3$

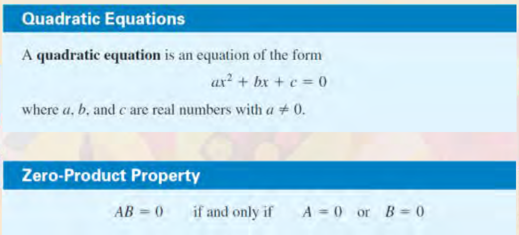
$\text{LHS} = 7(3) - 4$
$= 17$

$x = 3$

$\text{RHS} = 3(3) + 8$
$= 17$

LHS = RHS ✓

Quadratic Equations



Quadratic Equations

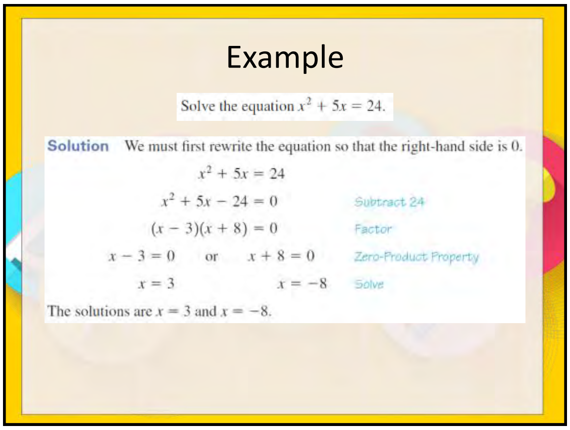
A **quadratic equation** is an equation of the form

$$ax^2 + bx + c = 0$$

where a, b, and c are real numbers with $a \neq 0$.

Zero-Product Property

$$AB = 0 \quad \text{if and only if} \quad A = 0 \quad \text{or} \quad B = 0$$



Example

Solve the equation $x^2 + 5x = 24$.

Solution We must first rewrite the equation so that the right-hand side is 0.

$x^2 + 5x = 24$

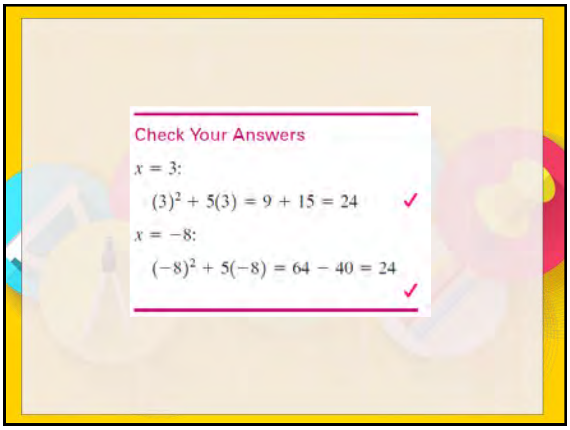
$x^2 + 5x - 24 = 0$ Subtract 24

$(x - 3)(x + 8) = 0$ Factor

$x - 3 = 0$ or $x + 8 = 0$ Zero-Product Property

$x = 3$ $x = -8$ Solve

The solutions are $x = 3$ and $x = -8$.



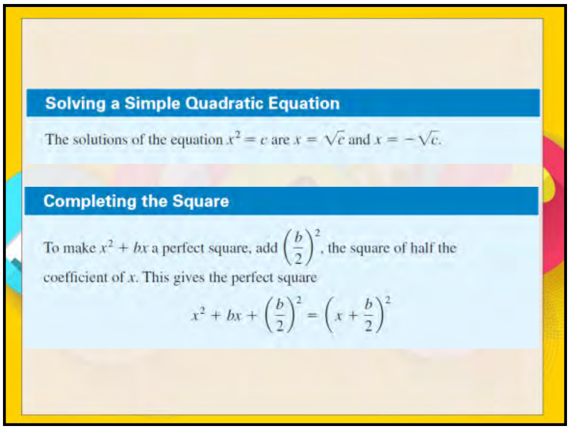
Check Your Answers

$x = 3$:

$(3)^2 + 5(3) = 9 + 15 = 24$ ✓

$x = -8$:

$(-8)^2 + 5(-8) = 64 - 40 = 24$ ✓



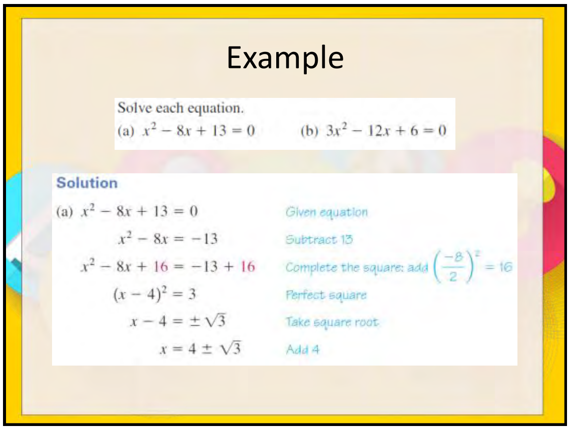
Solving a Simple Quadratic Equation

The solutions of the equation $x^2 = c$ are $x = \sqrt{c}$ and $x = -\sqrt{c}$.

Completing the Square

To make $x^2 + bx$ a perfect square, add $\left(\frac{b}{2}\right)^2$, the square of half the coefficient of x. This gives the perfect square

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$



Example

Solve each equation.

(a) $x^2 - 8x + 13 = 0$ (b) $3x^2 - 12x + 6 = 0$

Solution

(a) $x^2 - 8x + 13 = 0$ Given equation

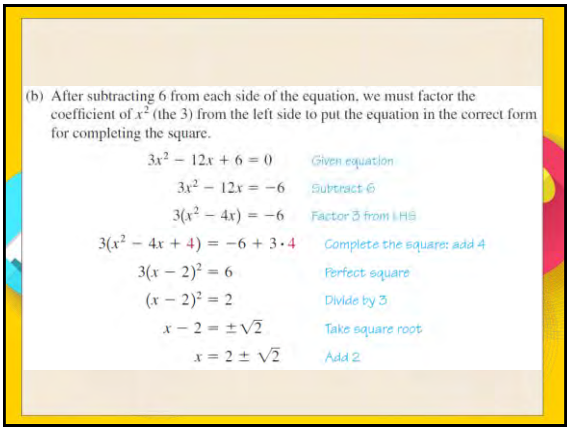
$x^2 - 8x = -13$ Subtract 13

$x^2 - 8x + 16 = -13 + 16$ Complete the square: add $\left(\frac{-8}{2}\right)^2 = 16$

$(x - 4)^2 = 3$ Perfect square

$x - 4 = \pm\sqrt{3}$ Take square root

$x = 4 \pm \sqrt{3}$ Add 4



(b) After subtracting 6 from each side of the equation, we must factor the coefficient of x^2 (the 3) from the left side to put the equation in the correct form for completing the square.

$3x^2 - 12x + 6 = 0$ Given equation

$3x^2 - 12x = -6$ Subtract 6

$3(x^2 - 4x) = -6$ Factor 3 from LHS

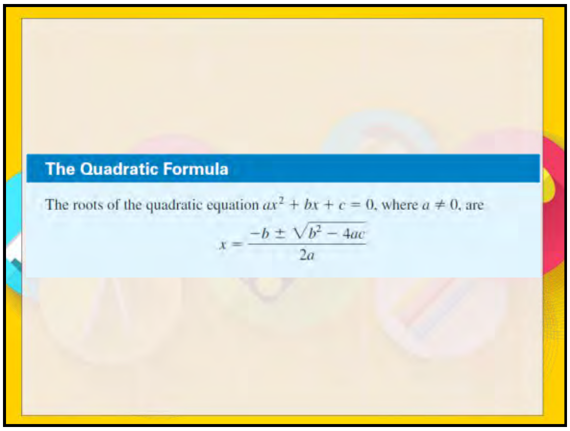
$3(x^2 - 4x + 4) = -6 + 3 \cdot 4$ Complete the square: add 4

$3(x - 2)^2 = 6$ Perfect square

$(x - 2)^2 = 2$ Divide by 3

$x - 2 = \pm\sqrt{2}$ Take square root

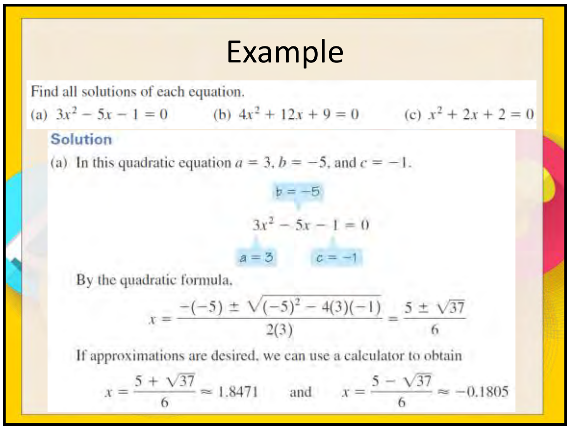
$x = 2 \pm \sqrt{2}$ Add 2



The Quadratic Formula

The roots of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$



Example

Find all solutions of each equation.

(a) $3x^2 - 5x - 1 = 0$ (b) $4x^2 + 12x + 9 = 0$ (c) $x^2 + 2x + 2 = 0$

Solution

(a) In this quadratic equation $a = 3$, $b = -5$, and $c = -1$.

$$\overset{b = -5}{3x^2 - 5x - 1 = 0}$$

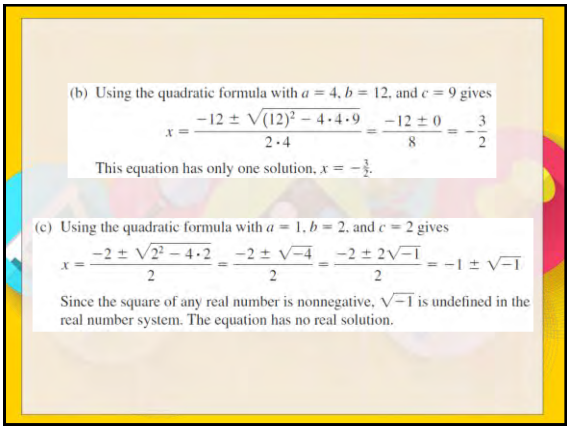
$a = 3$ $c = -1$

By the quadratic formula,

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(-1)}}{2(3)} = \frac{5 \pm \sqrt{37}}{6}$$

If approximations are desired, we can use a calculator to obtain

$$x = \frac{5 + \sqrt{37}}{6} \approx 1.8471 \quad \text{and} \quad x = \frac{5 - \sqrt{37}}{6} \approx -0.1805$$



(b) Using the quadratic formula with $a = 4$, $b = 12$, and $c = 9$ gives

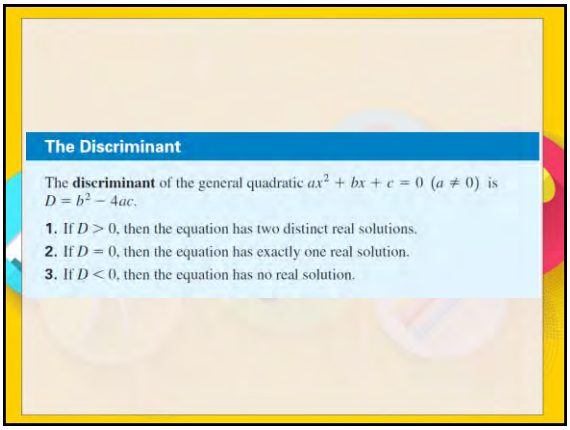
$$x = \frac{-12 \pm \sqrt{(12)^2 - 4 \cdot 4 \cdot 9}}{2 \cdot 4} = \frac{-12 \pm 0}{8} = -\frac{3}{2}$$

This equation has only one solution, $x = -\frac{3}{2}$.

(c) Using the quadratic formula with $a = 1$, $b = 2$, and $c = 2$ gives

$$x = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 2}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2\sqrt{-1}}{2} = -1 \pm \sqrt{-1}$$

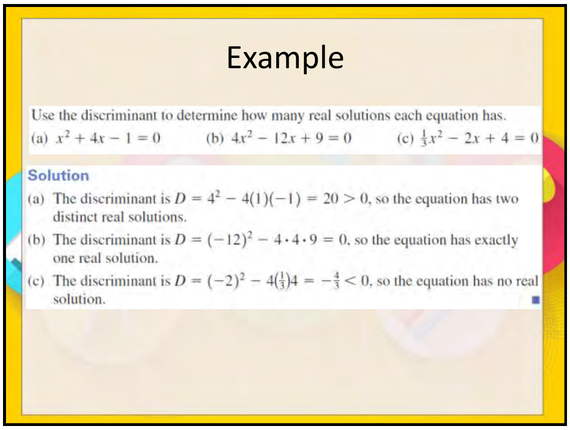
Since the square of any real number is nonnegative, $\sqrt{-1}$ is undefined in the real number system. The equation has no real solution.



The Discriminant

The **discriminant** of the general quadratic $ax^2 + bx + c = 0$ $(a \neq 0)$ is $D = b^2 - 4ac$.

1. If $D > 0$, then the equation has two distinct real solutions.
2. If $D = 0$, then the equation has exactly one real solution.
3. If $D < 0$, then the equation has no real solution.



Example

Use the discriminant to determine how many real solutions each equation has.

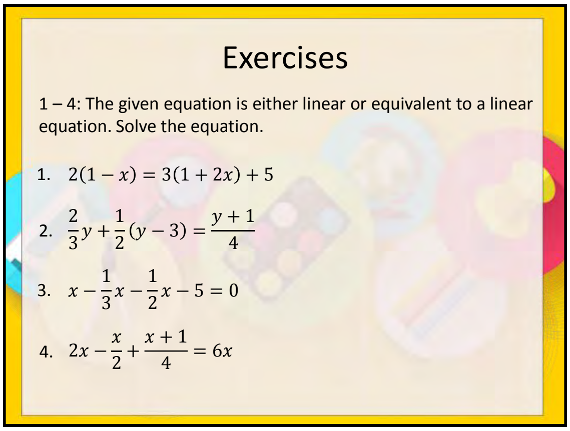
(a) $x^2 + 4x - 1 = 0$ (b) $4x^2 - 12x + 9 = 0$ (c) $\frac{1}{3}x^2 - 2x + 4 = 0$

Solution

(a) The discriminant is $D = 4^2 - 4(1)(-1) = 20 > 0$, so the equation has two distinct real solutions.

(b) The discriminant is $D = (-12)^2 - 4 \cdot 4 \cdot 9 = 0$, so the equation has exactly one real solution.

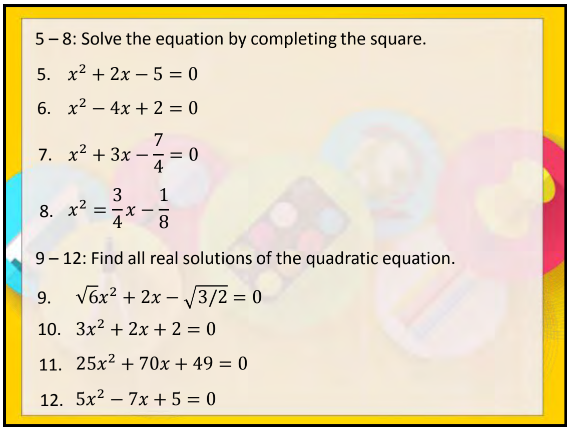
(c) The discriminant is $D = (-2)^2 - 4(\frac{1}{3})4 = -\frac{4}{3} < 0$, so the equation has no real solution. ■



Exercises

1 – 4: The given equation is either linear or equivalent to a linear equation. Solve the equation.

1. $2(1 - x) = 3(1 + 2x) + 5$
2. $\frac{2}{3}y + \frac{1}{2}(y - 3) = \frac{y + 1}{4}$
3. $x - \frac{1}{3}x - \frac{1}{2}x - 5 = 0$
4. $2x - \frac{x}{2} + \frac{x + 1}{4} = 6x$



5 – 8: Solve the equation by completing the square.

5. $x^2 + 2x - 5 = 0$
6. $x^2 - 4x + 2 = 0$
7. $x^2 + 3x - \frac{7}{4} = 0$
8. $x^2 = \frac{3}{4}x - \frac{1}{8}$

9 – 12: Find all real solutions of the quadratic equation.

9. $\sqrt{6}x^2 + 2x - \sqrt{3/2} = 0$
10. $3x^2 + 2x + 2 = 0$
11. $25x^2 + 70x + 49 = 0$
12. $5x^2 - 7x + 5 = 0$

Exponential Equations

An *exponential equation* is one in which the variable occurs in the exponent.

Sometimes we use the Laws of Logarithms to "bring down *x*" from the exponent.

For example

$$\begin{aligned} 2^x &= 7 && \text{Given equation} \\ \ln 2^x &= \ln 7 && \text{Take ln of each side} \\ x \ln 2 &= \ln 7 && \text{Law 3 (bring down exponent)} \\ x &= \frac{\ln 7}{\ln 2} && \text{Solve for } x \\ &\approx 2.807 && \text{Calculator} \end{aligned}$$

Guidelines for Solving Exponential Equations

1. Isolate the exponential expression on one side of the equation.
2. Take the logarithm of each side, then use the Laws of Logarithms to "bring down the exponent."
3. Solve for the variable.

Example

Find the solution of the equation $3^{x+2} = 7$, correct to six decimal places.

Solution We take the common logarithm of each side and use Law 3.

$$\begin{aligned} 3^{x+2} &= 7 && \text{Given equation} \\ \log(3^{x+2}) &= \log 7 && \text{Take log of each side} \\ (x + 2)\log 3 &= \log 7 && \text{Law 3 (bring down exponent)} \\ x + 2 &= \frac{\log 7}{\log 3} && \text{Divide by log 3} \\ x &= \frac{\log 7}{\log 3} - 2 && \text{Subtract 2} \\ &\approx -0.228756 && \text{Calculator} \end{aligned}$$

Example

Solve the equation $e^{3-2x} = 4$ algebraically and graphically.

Solution

Since the base of the exponential term is *e*, we use natural logarithms to solve this equation.

$$\begin{aligned} e^{3-2x} &= 4 && \text{Given equation} \\ \ln(e^{3-2x}) &= \ln 4 && \text{Take ln of each side} \\ 3 - 2x &= \ln 4 && \text{Property of ln} \\ 2x &= 3 - \ln 4 && \\ x &= \tfrac{1}{2}(3 - \ln 4) \approx 0.807 && \end{aligned}$$

You should check that this answer satisfies the original equation.

Example

Solve the equation $e^{2x} - e^x - 6 = 0$.

Solution To isolate the exponential term, we factor.

$$\begin{aligned} e^{2x} - e^x - 6 &= 0 && \text{Given equation} \\ (e^x)^2 - e^x - 6 &= 0 && \text{Law of Exponents} \\ (e^x - 3)(e^x + 2) &= 0 && \text{Factor (a quadratic in } e^x\text{)} \\ e^x - 3 = 0 \quad \text{or} \quad e^x + 2 &= 0 && \text{Zero-Product Property} \\ e^x = 3 \qquad\qquad e^x &= -2 && \end{aligned}$$

The equation $e^x = 3$ leads to $x = \ln 3$. But the equation $e^x = -2$ has no solution because $e^x > 0$ for all x. Thus, $x = \ln 3 \approx 1.0986$ is the only solution. You should check that this answer satisfies the original equation.

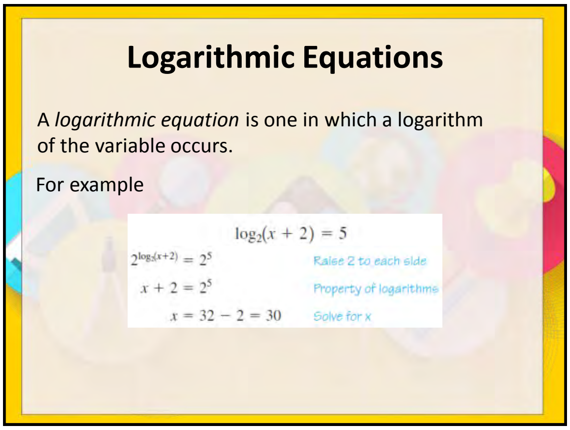
Example

Solve the equation $3xe^x + x^2e^x = 0$.

Solution First we factor the left side of the equation.

$$\begin{aligned} 3xe^x + x^2e^x &= 0 && \text{Given equation} \\ x(3 + x)e^x &= 0 && \text{Factor out common factors} \\ x(3 + x) &= 0 && \text{Divide by } e^x \text{ (because } e^x \neq 0\text{)} \\ x = 0 \quad \text{or} \quad 3 + x &= 0 && \text{Zero-Product Property} \end{aligned}$$

Thus, the solutions are $x = 0$ and $x = -3$.



Logarithmic Equations

A *logarithmic equation* is one in which a logarithm of the variable occurs.

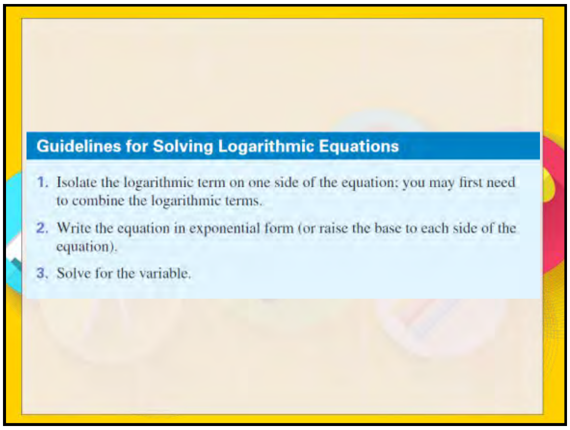
For example

$\log_2(x+2) = 5$

$2^{\log_2(x+2)} = 2^5$ Raise 2 to each side

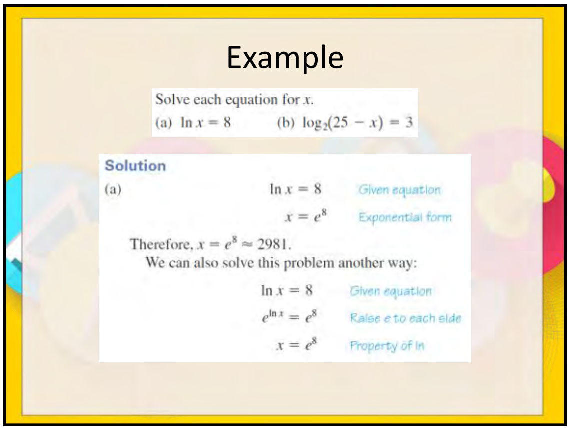
$x + 2 = 2^5$ Property of logarithms

$x = 32 - 2 = 30$ Solve for x



Guidelines for Solving Logarithmic Equations

1. Isolate the logarithmic term on one side of the equation; you may first need to combine the logarithmic terms.
2. Write the equation in exponential form (or raise the base to each side of the equation).
3. Solve for the variable.



Example

Solve each equation for x.

(a) $\ln x = 8$ (b) $\log_2(25 - x) = 3$

Solution

(a) $\ln x = 8$ Given equation

$x = e^8$ Exponential form

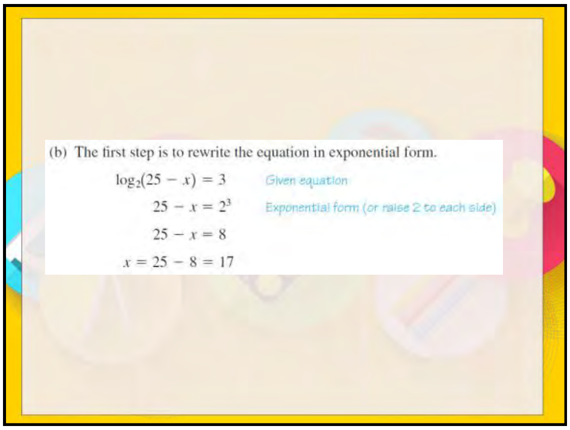
Therefore, $x = e^8 \approx 2981$.

We can also solve this problem another way:

$\ln x = 8$ Given equation

$e^{\ln x} = e^8$ Raise e to each side

$x = e^8$ Property of ln



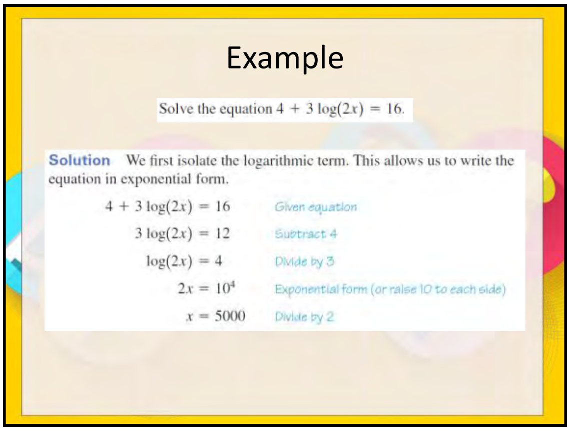
(b) The first step is to rewrite the equation in exponential form.

$\log_2(25 - x) = 3$ Given equation

$25 - x = 2^3$ Exponential form (or raise 2 to each side)

$25 - x = 8$

$x = 25 - 8 = 17$



Example

Solve the equation $4 + 3\log(2x) = 16$.

Solution We first isolate the logarithmic term. This allows us to write the equation in exponential form.

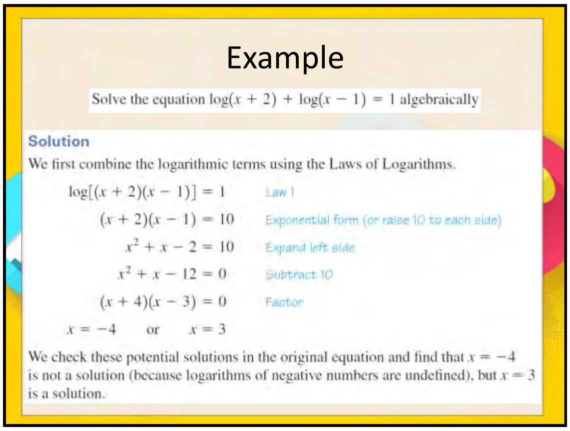
$4 + 3\log(2x) = 16$ Given equation

$3\log(2x) = 12$ Subtract 4

$\log(2x) = 4$ Divide by 3

$2x = 10^4$ Exponential form (or raise 10 to each side)

$x = 5000$ Divide by 2



Example

Solve the equation $\log(x+2) + \log(x-1) = 1$ algebraically

Solution

We first combine the logarithmic terms using the Laws of Logarithms.

$\log[(x+2)(x-1)] = 1$ Law 1

$(x+2)(x-1) = 10$ Exponential form (or raise 10 to each side)

$x^2 + x - 2 = 10$ Expand left side

$x^2 + x - 12 = 0$ Subtract 10

$(x+4)(x-3) = 0$ Factor

$x = -4$ or $x = 3$

We check these potential solutions in the original equation and find that $x = -4$ is not a solution (because logarithms of negative numbers are undefined), but $x = 3$ is a solution.

Check Your Answers

$x = -4$:

$\log(-4+2) + \log(-4-1)$

$= \log(-2) + \log(-5)$

undefined ✗

$x = 3$:

$\log(3+2) + \log(3-1)$

$= \log 5 + \log 2 = \log(5 \cdot 2)$

$= \log 10 = 1$ ✓

Exercises

13 – 16: Solve the equation.

13. $x^2 2^x - 2^x = 0$
14. $x^2 10^x - x10^x = 2(10^x)$
15. $4x^3 e^{-3x} - 3x^4 e^{-3x} = 0$
16. $x^2 e^x + xe^x - e^x = 0$

17 – 19: Solve the logarithmic equation for x.

17. $\log_5 x + \log_5(x+1) = \log_5 20$
18. $\log_5(x+1) - \log_5(x-1) = 2$
19. $\log x + \log(x-3) = 1$

Inequalities

An inequality looks just like an equation, except that in the place of the equal sign is one of the symbols, <, >, ≤, or ≥. Here is an example of an inequality:

The following illustration shows how an inequality differs from its corresponding equation:

		Solution	Graph
Equation:	$4x + 7 = 19$	$x = 3$	
Inequality:	$4x + 7 \le 19$	$x \le 3$	

Linear Inequalities

An inequality is **linear** if each term is constant or a multiple of the variable.

Example

Solve the inequality $3x < 9x + 4$ and sketch the solution set.

Solution

$$3x < 9x + 4$$

$$3x - 9x < 9x + 4 - 9x \quad \text{Subtract } 9x$$

$$-6x < 4 \quad \text{Simplify}$$

$$(-\tfrac{1}{6})(-6x) > (-\tfrac{1}{6})(4) \quad \text{Multiply by } -\tfrac{1}{6} \text{ (or divide by } -6\text{)}$$

$$x > -\tfrac{2}{3} \quad \text{Simplify}$$

The solution set consists of all numbers greater than $-\frac{2}{3}$. In other words the solution of the inequality is the interval $(-\frac{2}{3}, \infty)$.

Example

Solve the inequalities $4 \le 3x - 2 < 13$.

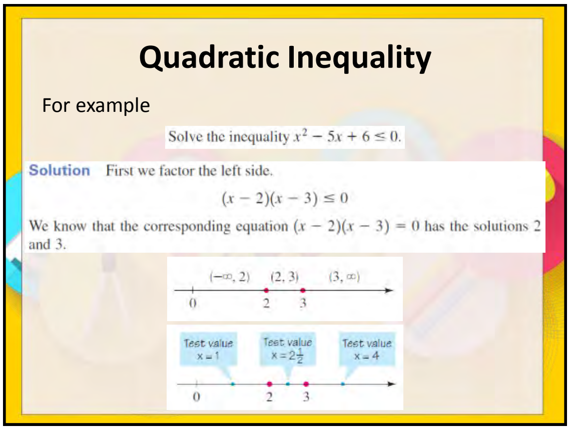
Solution The solution set consists of all values of x that satisfy both of the inequalities $4 \le 3x - 2$ and $3x - 2 < 13$. Using Rules 1 and 3, we see that the following inequalities are equivalent:

$$4 \le 3x - 2 < 13$$

$$6 \le 3x < 15 \quad \text{Add 2}$$

$$2 \le x < 5 \quad \text{Divide by 3}$$

Therefore, the solution set is $[2, 5)$, as shown in Figure 2.



Quadratic Inequality

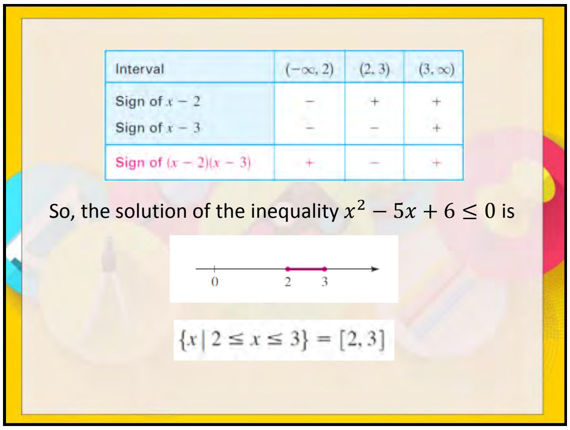
For example

Solve the inequality $x^2 - 5x + 6 \le 0$.

Solution First we factor the left side.

$$(x-2)(x-3) \le 0$$

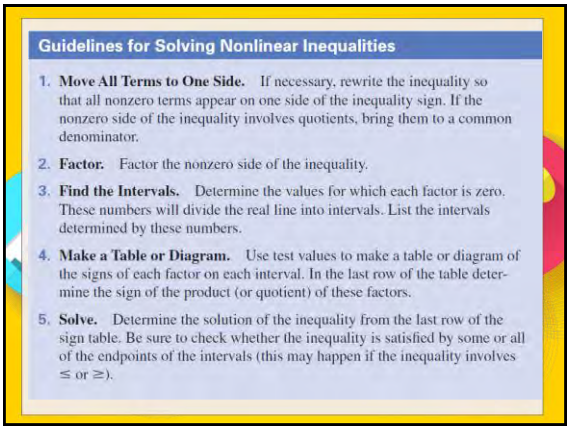
We know that the corresponding equation $(x-2)(x-3) = 0$ has the solutions 2 and 3.



Interval	$(-\infty, 2)$	$(2, 3)$	$(3, \infty)$
Sign of $x - 2$	$-$	$+$	$+$
Sign of $x - 3$	$-$	$-$	$+$
Sign of $(x-2)(x-3)$	$+$	$-$	$+$

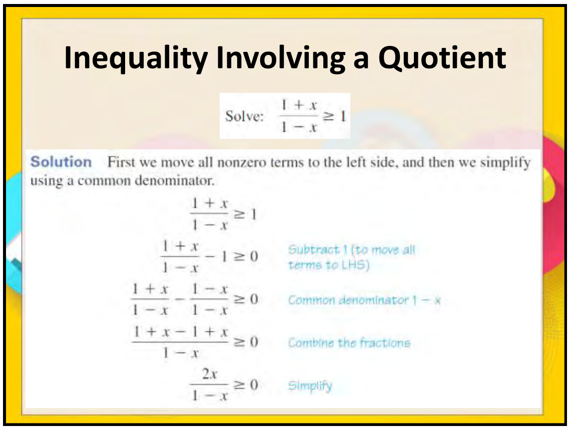
So, the solution of the inequality $x^2 - 5x + 6 \le 0$ is

$$\{x \mid 2 \le x \le 3\} = [2, 3]$$



Guidelines for Solving Nonlinear Inequalities

1. **Move All Terms to One Side.** If necessary, rewrite the inequality so that all nonzero terms appear on one side of the inequality sign. If the nonzero side of the inequality involves quotients, bring them to a common denominator.
2. **Factor.** Factor the nonzero side of the inequality.
3. **Find the Intervals.** Determine the values for which each factor is zero. These numbers will divide the real line into intervals. List the intervals determined by these numbers.
4. **Make a Table or Diagram.** Use test values to make a table or diagram of the signs of each factor on each interval. In the last row of the table determine the sign of the product (or quotient) of these factors.
5. **Solve.** Determine the solution of the inequality from the last row of the sign table. Be sure to check whether the inequality is satisfied by some or all of the endpoints of the intervals (this may happen if the inequality involves $\le$ or $\ge$).



Inequality Involving a Quotient

Solve: $\dfrac{1+x}{1-x} \ge 1$

Solution First we move all nonzero terms to the left side, and then we simplify using a common denominator.

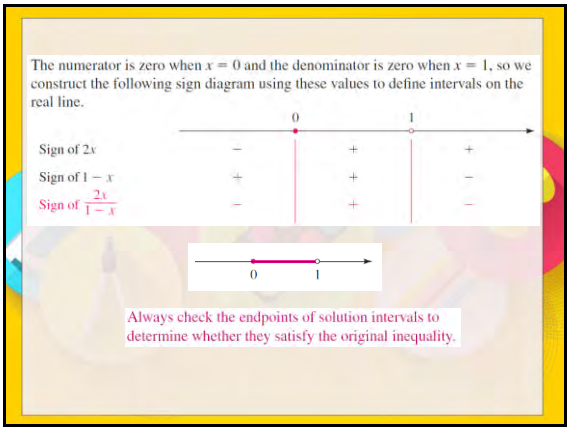
$$\frac{1+x}{1-x} \ge 1$$

$$\frac{1+x}{1-x} - 1 \ge 0 \qquad \text{Subtract 1 (to move all terms to LHS)}$$

$$\frac{1+x}{1-x} - \frac{1-x}{1-x} \ge 0 \qquad \text{Common denominator } 1 - x$$

$$\frac{1+x-1+x}{1-x} \ge 0 \qquad \text{Combine the fractions}$$

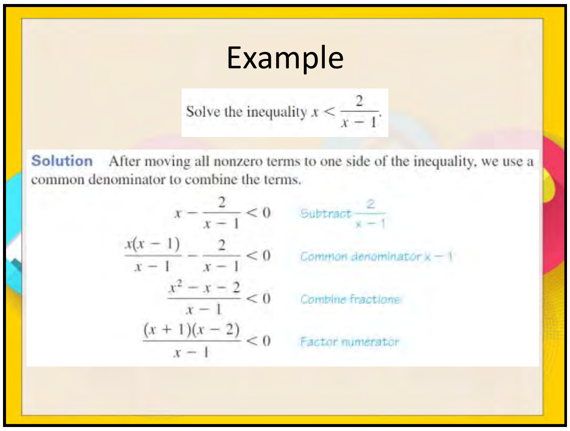
$$\frac{2x}{1-x} \ge 0 \qquad \text{Simplify}$$



The numerator is zero when $x = 0$ and the denominator is zero when $x = 1$, so we construct the following sign diagram using these values to define intervals on the real line.

		0		1	
Sign of $2x$	$-$		$+$		$+$
Sign of $1 - x$	$+$		$+$		$-$
Sign of $\frac{2x}{1-x}$	$-$		$+$		$-$

Always check the endpoints of solution intervals to determine whether they satisfy the original inequality.



Example

Solve the inequality $x < \dfrac{2}{x-1}$.

Solution After moving all nonzero terms to one side of the inequality, we use a common denominator to combine the terms.

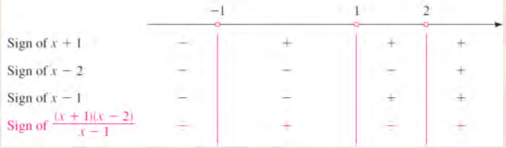
$$x - \frac{2}{x-1} < 0 \qquad \text{Subtract } \frac{2}{x-1}$$

$$\frac{x(x-1)}{x-1} - \frac{2}{x-1} < 0 \qquad \text{Common denominator } x - 1$$

$$\frac{x^2 - x - 2}{x-1} < 0 \qquad \text{Combine fractions}$$

$$\frac{(x+1)(x-2)}{x-1} < 0 \qquad \text{Factor numerator}$$

The factors in this quotient change sign at -1, 1, and 2, so we must examine the intervals $(-\infty, -1)$, $(-1, 1)$, $(1, 2)$, and $(2, \infty)$. Using test values, we get the following sign diagram.



	$(-\infty,-1)$	$(-1,1)$	$(1,2)$	$(2,\infty)$
Sign of $x + 1$	−	+	+	+
Sign of $x - 2$	−	−	−	+
Sign of $x - 1$	−	−	+	+
Sign of $\frac{(x+1)(x-2)}{x-1}$	−	+	−	+

Since the quotient must be negative, the solution is

$$(-\infty, -1) \cup (1, 2)$$

Exercises

20 – 23: Solve the linear inequality. Express the solution using interval notation and graph the solution set.

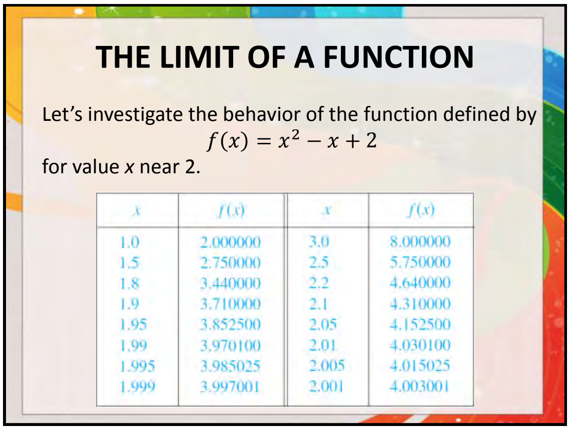
20. $4 - 3x \le -(1 + 8x)$
21. $2(7x - 3) \le 12x + 16$
22. $2 \le x + 5 < 4$
23. $5 \le 3x - 4 \le 14$

24 – 29: Solve the nonlinear inequality. Express the solution using interval notation and graph the solution set.

24. $-2x \le 4$
25. $(x + 2)(x - 1)(x - 3) \le 0$
26. $x^3 - 4x > 0$
27. $16x \le x^3$
28. $\dfrac{x - 3}{x + 1} \ge 0$
29. $\dfrac{2x + 6}{x - 2} < 0$

Meeting 5

Derivative - 1



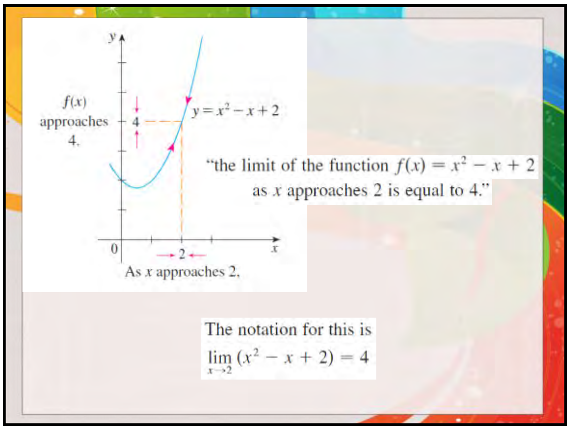
THE LIMIT OF A FUNCTION

Let's investigate the behavior of the function defined by

$$f(x) = x^2 - x + 2$$

for value *x* near 2.

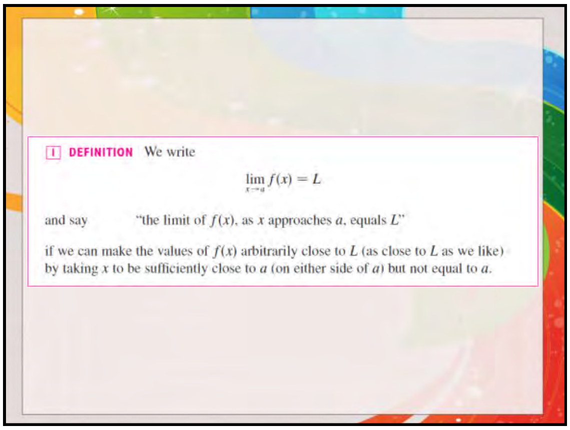
x	$f(x)$	x	$f(x)$
1.0	2.000000	3.0	8.000000
1.5	2.750000	2.5	5.750000
1.8	3.440000	2.2	4.640000
1.9	3.710000	2.1	4.310000
1.95	3.852500	2.05	4.152500
1.99	3.970100	2.01	4.030100
1.995	3.985025	2.005	4.015025
1.999	3.997001	2.001	4.003001



"the limit of the function $f(x) = x^2 - x + 2$ as x approaches 2 is equal to 4."

The notation for this is

$$\lim_{x \to 2} (x^2 - x + 2) = 4$$

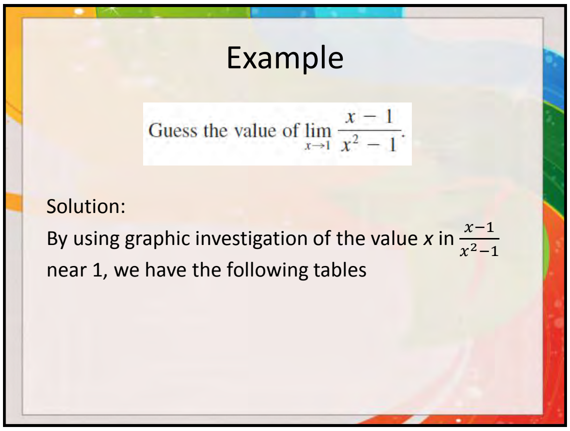


☐1 **DEFINITION** We write

$$\lim_{x \to a} f(x) = L$$

and say "the limit of $f(x)$, as x approaches a, equals L"

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a (on either side of a) but not equal to a.

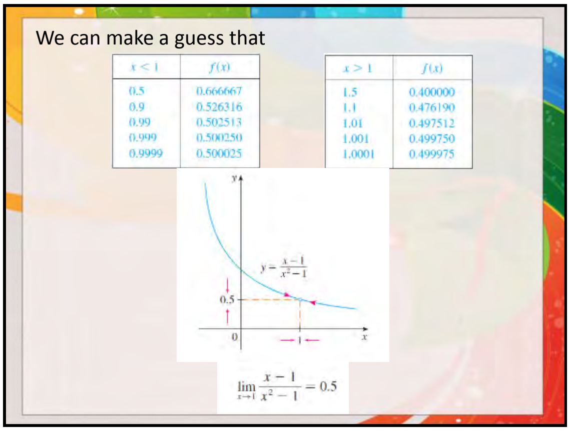


Example

Guess the value of $\lim_{x \to 1} \dfrac{x - 1}{x^2 - 1}$.

Solution:

By using graphic investigation of the value *x* in $\frac{x-1}{x^2-1}$ near 1, we have the following tables

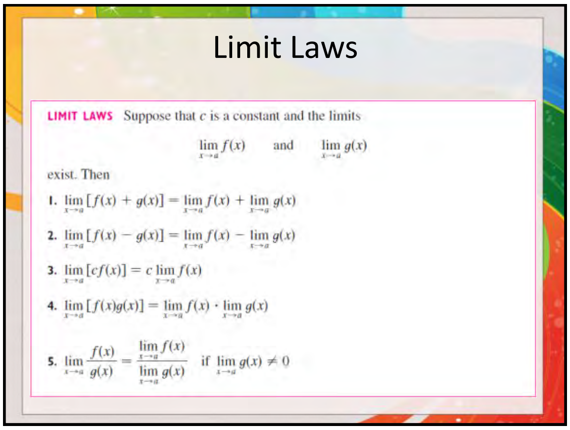


We can make a guess that

$x < 1$	$f(x)$
0.5	0.666667
0.9	0.526316
0.99	0.502513
0.999	0.500250
0.9999	0.500025

$x > 1$	$f(x)$
1.5	0.400000
1.1	0.476190
1.01	0.497512
1.001	0.499750
1.0001	0.499975

$$\lim_{x \to 1} \frac{x - 1}{x^2 - 1} = 0.5$$



Limit Laws

LIMIT LAWS Suppose that c is a constant and the limits

$$\lim_{x \to a} f(x) \quad \text{and} \quad \lim_{x \to a} g(x)$$

exist. Then

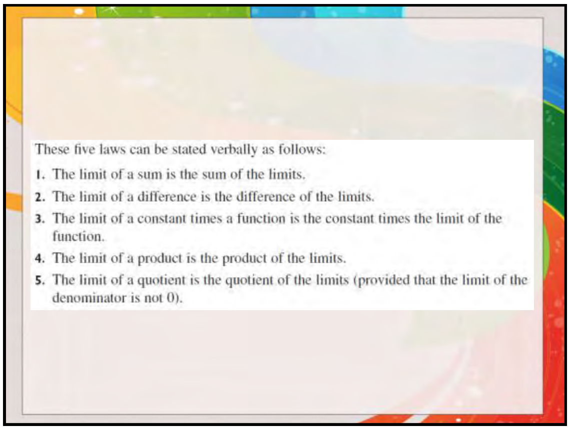
1. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

2. $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

3. $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$

4. $\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

5. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ if $\lim_{x \to a} g(x) \neq 0$



These five laws can be stated verbally as follows:

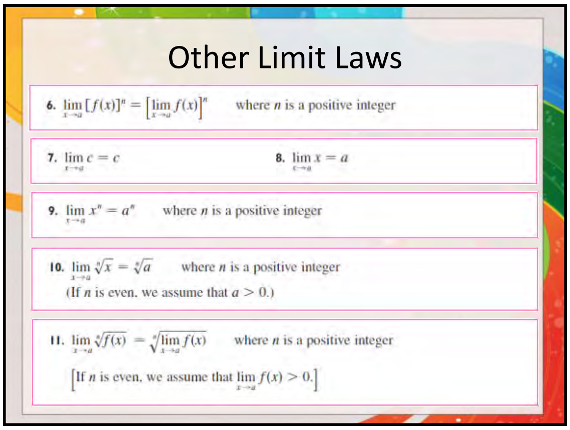
1. The limit of a sum is the sum of the limits.

2. The limit of a difference is the difference of the limits.

3. The limit of a constant times a function is the constant times the limit of the function.

4. The limit of a product is the product of the limits.

5. The limit of a quotient is the quotient of the limits (provided that the limit of the denominator is not 0).



Other Limit Laws

6. $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$ where n is a positive integer

7. $\lim_{x \to a} c = c$ **8.** $\lim_{x \to a} x = a$

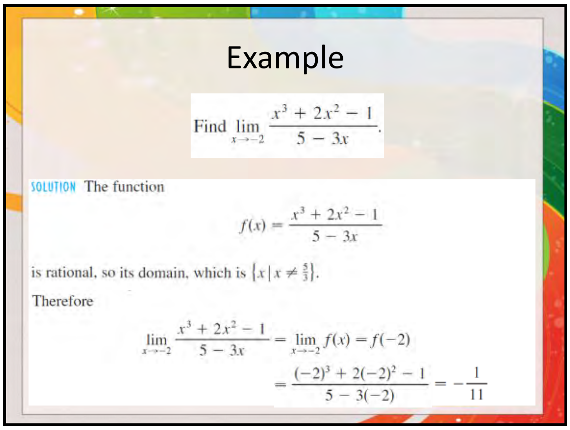
9. $\lim_{x \to a} x^n = a^n$ where n is a positive integer

10. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer

(If n is even, we assume that $a > 0$.)

11. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ where n is a positive integer

[If n is even, we assume that $\lim_{x \to a} f(x) > 0$.]



Example

Find $\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$.

SOLUTION The function

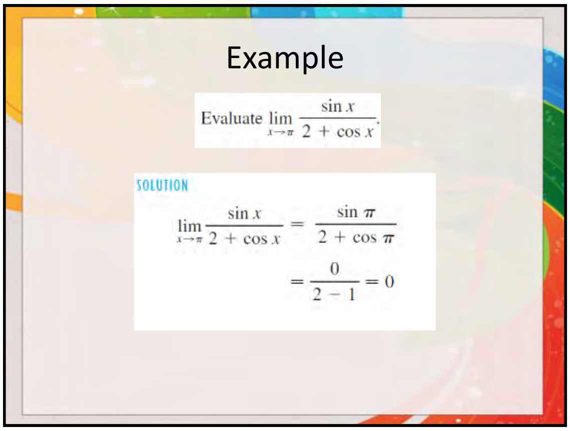
$$f(x) = \frac{x^3 + 2x^2 - 1}{5 - 3x}$$

is rational, so its domain, which is $\{x \mid x \neq \frac{5}{3}\}$.

Therefore

$$\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} = \lim_{x \to -2} f(x) = f(-2)$$

$$= \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)} = -\frac{1}{11}$$



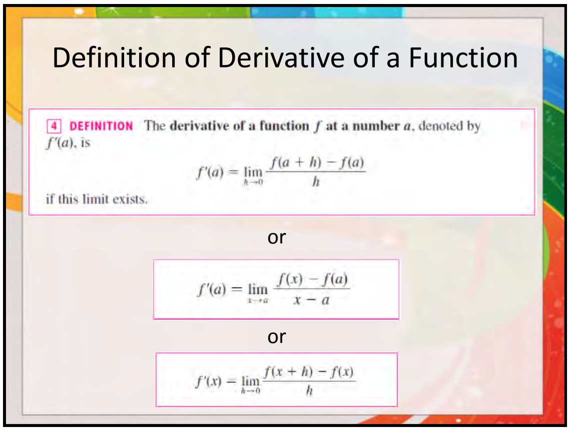
Example

Evaluate $\lim_{x \to \pi} \frac{\sin x}{2 + \cos x}$.

SOLUTION

$$\lim_{x \to \pi} \frac{\sin x}{2 + \cos x} = \frac{\sin \pi}{2 + \cos \pi}$$

$$= \frac{0}{2 - 1} = 0$$



Definition of Derivative of a Function

4 DEFINITION The **derivative of a function f at a number a**, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

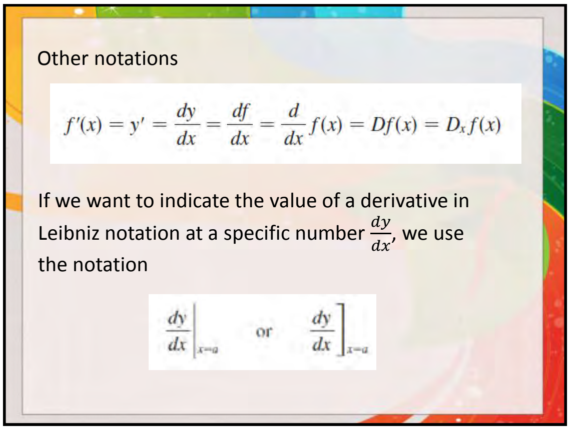
if this limit exists.

or

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

or

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

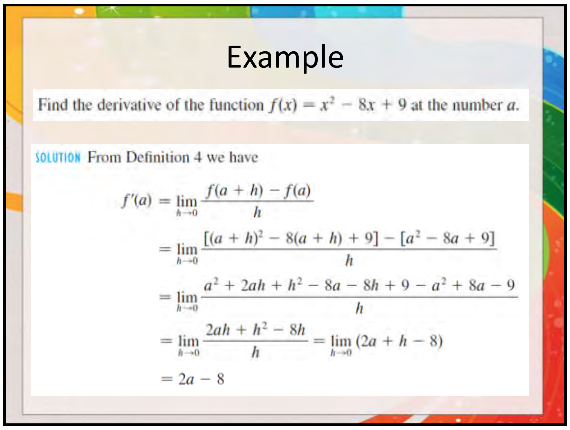


Other notations

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x f(x)$$

If we want to indicate the value of a derivative in Leibniz notation at a specific number $\frac{dy}{dx}$, we use the notation

$$\left.\frac{dy}{dx}\right|_{x=a} \quad \text{or} \quad \left.\frac{dy}{dx}\right]_{x=a}$$

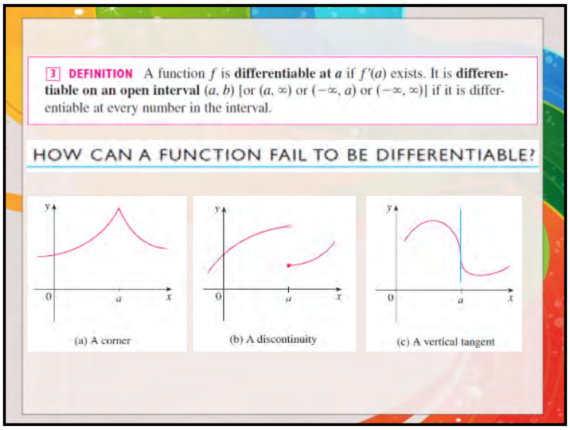


Example

Find the derivative of the function $f(x) = x^2 - 8x + 9$ at the number a.

SOLUTION From Definition 4 we have

$$\begin{aligned} f'(a) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h \to 0} \frac{[(a+h)^2 - 8(a+h) + 9] - [a^2 - 8a + 9]}{h} \\ &= \lim_{h \to 0} \frac{a^2 + 2ah + h^2 - 8a - 8h + 9 - a^2 + 8a - 9}{h} \\ &= \lim_{h \to 0} \frac{2ah + h^2 - 8h}{h} = \lim_{h \to 0} (2a + h - 8) \\ &= 2a - 8 \end{aligned}$$



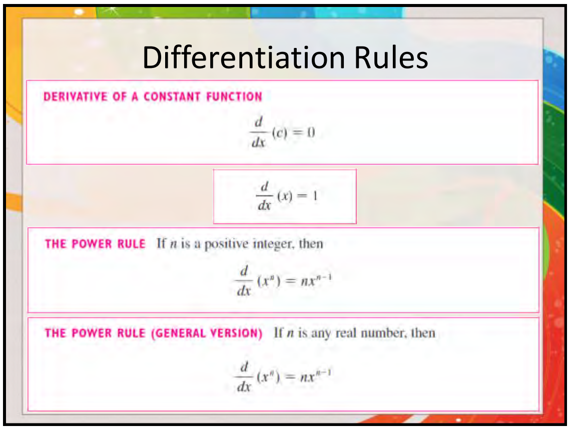
[3] DEFINITION A function f is **differentiable at a** if $f'(a)$ exists. It is **differentiable on an open interval** (a, b) [or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

HOW CAN A FUNCTION FAIL TO BE DIFFERENTIABLE?

(a) A corner

(b) A discontinuity

(c) A vertical tangent



Differentiation Rules

DERIVATIVE OF A CONSTANT FUNCTION

$$\frac{d}{dx}(c) = 0$$

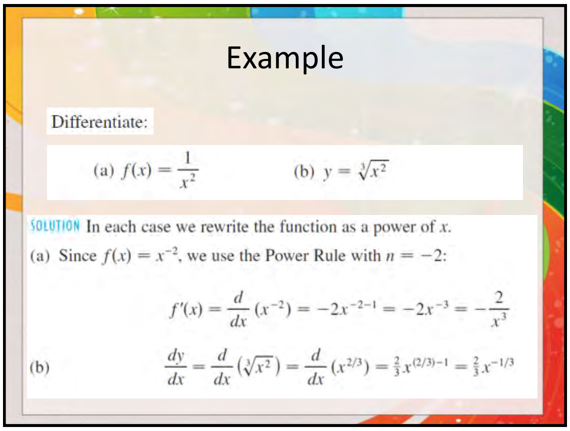
$$\frac{d}{dx}(x) = 1$$

THE POWER RULE If n is a positive integer, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

THE POWER RULE (GENERAL VERSION) If n is any real number, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$



Example

Differentiate:

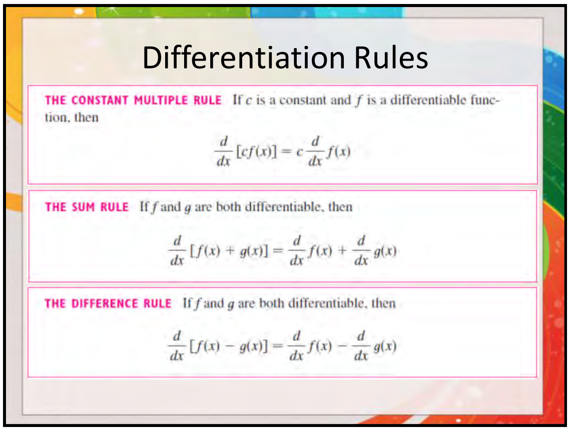
(a) $f(x) = \frac{1}{x^2}$ (b) $y = \sqrt[3]{x^2}$

SOLUTION In each case we rewrite the function as a power of x.

(a) Since $f(x) = x^{-2}$, we use the Power Rule with $n = -2$:

$$f'(x) = \frac{d}{dx}(x^{-2}) = -2x^{-2-1} = -2x^{-3} = -\frac{2}{x^3}$$

(b) $$\frac{dy}{dx} = \frac{d}{dx}(\sqrt[3]{x^2}) = \frac{d}{dx}(x^{2/3}) = \tfrac{2}{3}x^{(2/3)-1} = \tfrac{2}{3}x^{-1/3}$$



Differentiation Rules

THE CONSTANT MULTIPLE RULE If c is a constant and f is a differentiable function, then

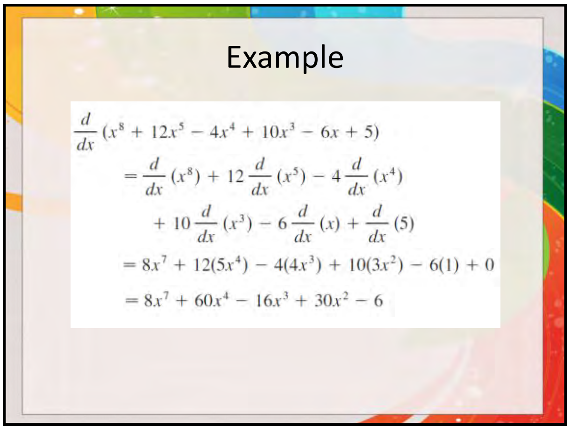
$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$$

THE SUM RULE If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

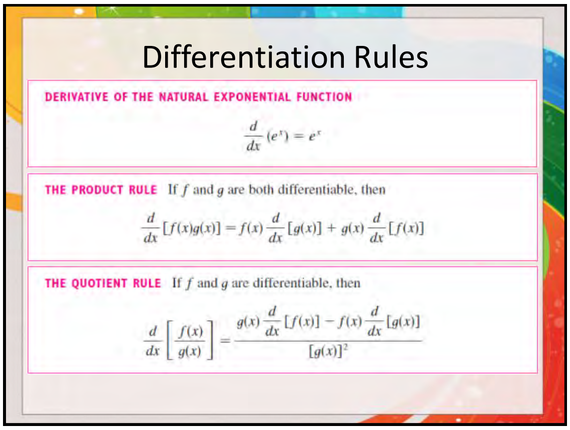
THE DIFFERENCE RULE If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$



Example

$$\frac{d}{dx}(x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + 5)$$
$$= \frac{d}{dx}(x^8) + 12\frac{d}{dx}(x^5) - 4\frac{d}{dx}(x^4)$$
$$+ 10\frac{d}{dx}(x^3) - 6\frac{d}{dx}(x) + \frac{d}{dx}(5)$$
$$= 8x^7 + 12(5x^4) - 4(4x^3) + 10(3x^2) - 6(1) + 0$$
$$= 8x^7 + 60x^4 - 16x^3 + 30x^2 - 6$$



Differentiation Rules

DERIVATIVE OF THE NATURAL EXPONENTIAL FUNCTION

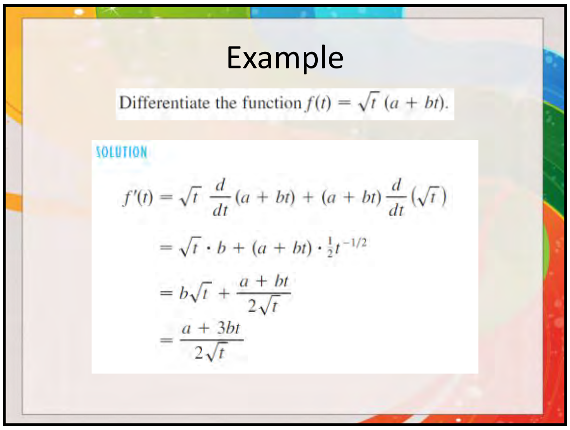
$$\frac{d}{dx}(e^x) = e^x$$

THE PRODUCT RULE If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

THE QUOTIENT RULE If f and g are differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

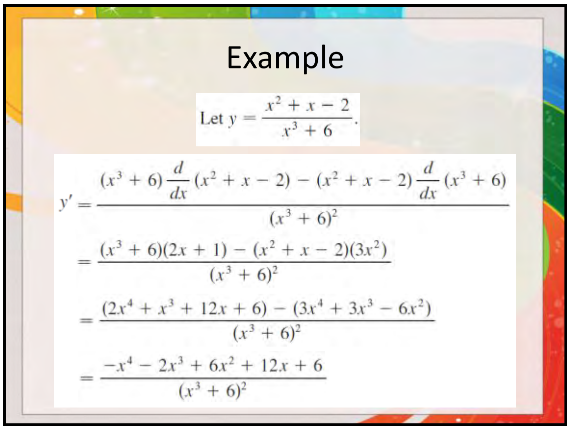


Example

Differentiate the function $f(t) = \sqrt{t}\,(a + bt)$.

SOLUTION

$$f'(t) = \sqrt{t}\,\frac{d}{dt}(a + bt) + (a + bt)\frac{d}{dt}(\sqrt{t})$$
$$= \sqrt{t} \cdot b + (a + bt) \cdot \tfrac{1}{2}t^{-1/2}$$
$$= b\sqrt{t} + \frac{a + bt}{2\sqrt{t}}$$
$$= \frac{a + 3bt}{2\sqrt{t}}$$



Example

Let $y = \dfrac{x^2 + x - 2}{x^3 + 6}$.

$$y' = \frac{(x^3 + 6)\frac{d}{dx}(x^2 + x - 2) - (x^2 + x - 2)\frac{d}{dx}(x^3 + 6)}{(x^3 + 6)^2}$$
$$= \frac{(x^3 + 6)(2x + 1) - (x^2 + x - 2)(3x^2)}{(x^3 + 6)^2}$$
$$= \frac{(2x^4 + x^3 + 12x + 6) - (3x^4 + 3x^3 - 6x^2)}{(x^3 + 6)^2}$$
$$= \frac{-x^4 - 2x^3 + 6x^2 + 12x + 6}{(x^3 + 6)^2}$$

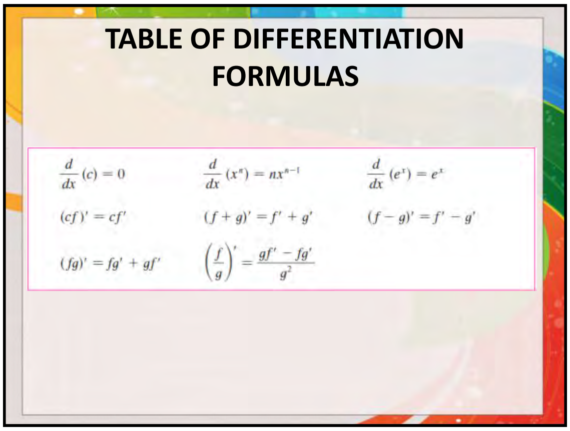
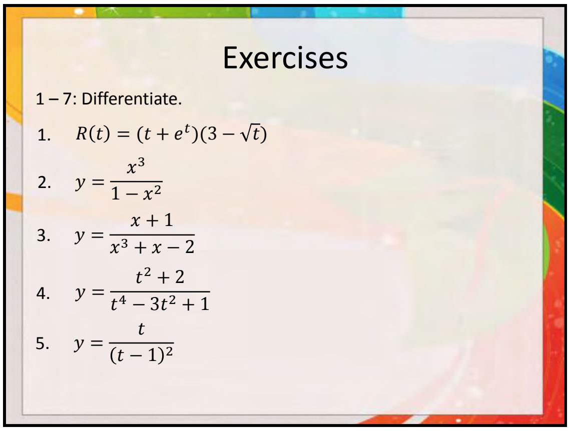


TABLE OF DIFFERENTIATION FORMULAS

$\frac{d}{dx}(c) = 0$	$\frac{d}{dx}(x^n) = nx^{n-1}$	$\frac{d}{dx}(e^x) = e^x$
$(cf)' = cf'$	$(f + g)' = f' + g'$	$(f - g)' = f' - g'$
$(fg)' = fg' + gf'$	$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$	



Exercises

1 – 7: Differentiate.

1. $R(t) = (t + e^t)(3 - \sqrt{t})$
2. $y = \dfrac{x^3}{1 - x^2}$
3. $y = \dfrac{x + 1}{x^3 + x - 2}$
4. $y = \dfrac{t^2 + 2}{t^4 - 3t^2 + 1}$
5. $y = \dfrac{t}{(t - 1)^2}$

6. $y = (r^2 - 2r)e^r$

7. $y = \dfrac{1}{s + ke^s}$

DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

$\frac{d}{dx}(\sin x) = \cos x$	$\frac{d}{dx}(\csc x) = -\csc x \cot x$
$\frac{d}{dx}(\cos x) = -\sin x$	$\frac{d}{dx}(\sec x) = \sec x \tan x$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\frac{d}{dx}(\cot x) = -\csc^2 x$

Example

Differentiate $f(x) = \dfrac{\sec x}{1 + \tan x}$.

SOLUTION

$$\begin{aligned} f'(x) &= \frac{(1 + \tan x)\dfrac{d}{dx}(\sec x) - \sec x\dfrac{d}{dx}(1 + \tan x)}{(1 + \tan x)^2} \\ &= \frac{(1 + \tan x)\sec x \tan x - \sec x \cdot \sec^2 x}{(1 + \tan x)^2} \\ &= \frac{\sec x(\tan x + \tan^2 x - \sec^2 x)}{(1 + \tan x)^2} \\ &= \frac{\sec x(\tan x - 1)}{(1 + \tan x)^2} \end{aligned}$$

Exercises

8 – 13: Differentiate.

8. $f(\theta) = \dfrac{\sec\theta}{1 + \sec\theta}$

9. $y = \dfrac{1 - \sec x}{\tan x}$

10. $y = \dfrac{\sin x}{x^2}$

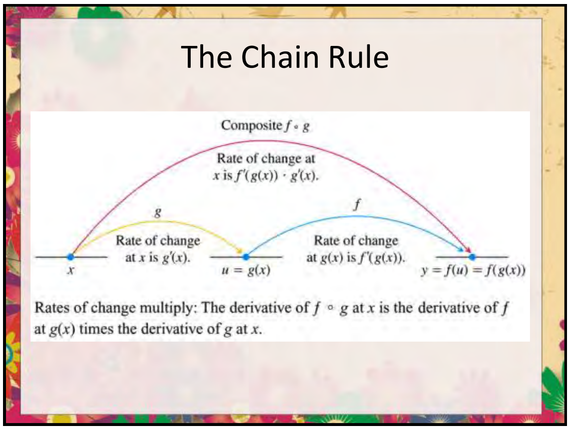
11. $y = \csc\theta\,(\theta + \cot\theta)$

12. $y = xe^x \csc x$

13. $y = x^2 \sin x \tan x$

Meeting 6

Derivative - 2



The Chain Rule

Composite $f \circ g$

Rate of change at x is $f'(g(x)) \cdot g'(x)$.

g

Rate of change at x is $g'(x)$.

f

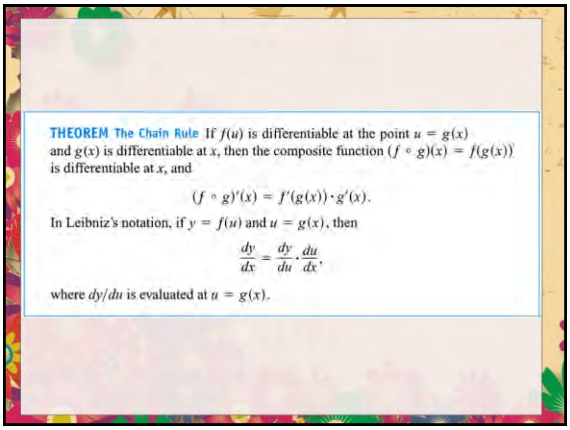
Rate of change at $g(x)$ is $f'(g(x))$.

x

$u = g(x)$

$y = f(u) = f(g(x))$

Rates of change multiply: The derivative of $f \circ g$ at x is the derivative of f at $g(x)$ times the derivative of g at x.



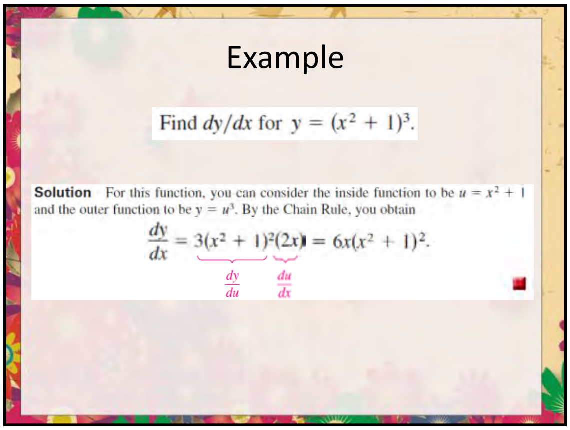
THEOREM The Chain Rule If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

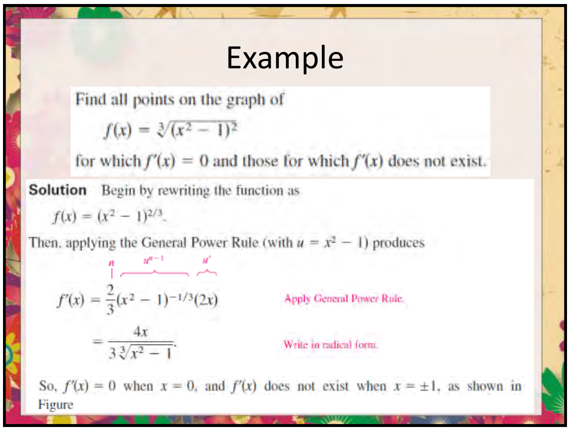


Example

Find dy/dx for $y = (x^2 + 1)^3$.

Solution For this function, you can consider the inside function to be $u = x^2 + 1$ and the outer function to be $y = u^3$. By the Chain Rule, you obtain

$$\frac{dy}{dx} = \underbrace{3(x^2 + 1)^2}_{\frac{dy}{du}}\underbrace{(2x)}_{\frac{du}{dx}} = 6x(x^2 + 1)^2.$$



Example

Find all points on the graph of

$$f(x) = \sqrt[3]{(x^2 - 1)^2}$$

for which $f'(x) = 0$ and those for which $f'(x)$ does not exist.

Solution Begin by rewriting the function as

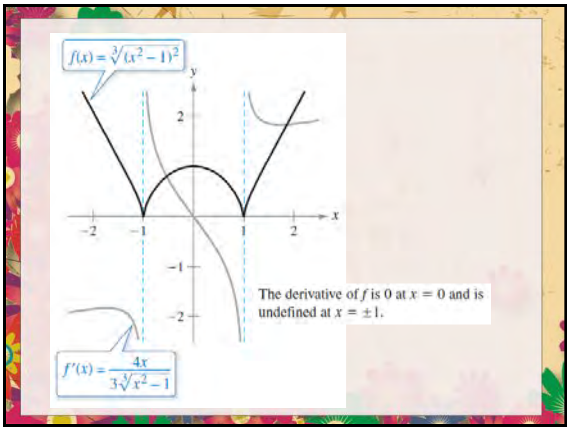
$$f(x) = (x^2 - 1)^{2/3}.$$

Then, applying the General Power Rule (with $u = x^2 - 1$) produces

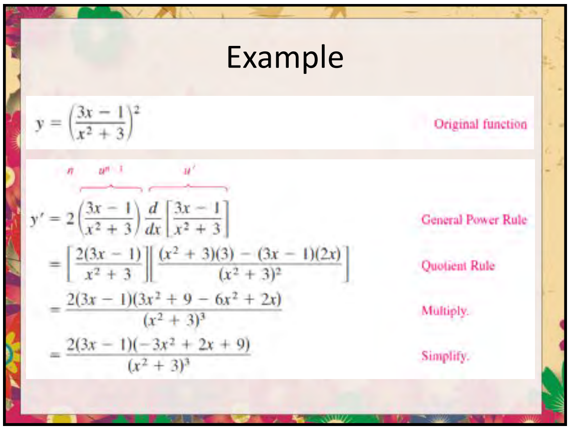
$$f'(x) = \overset{n}{\frac{2}{3}}\overbrace{(x^2 - 1)^{-1/3}}^{u^{n-1}}\overbrace{(2x)}^{u'} \qquad \text{Apply General Power Rule.}$$

$$= \frac{4x}{3\sqrt[3]{x^2 - 1}}. \qquad \text{Write in radical form.}$$

So, $f'(x) = 0$ when $x = 0$, and $f'(x)$ does not exist when $x = \pm 1$, as shown in Figure



The derivative of f is 0 at $x = 0$ and is undefined at $x = \pm 1$.



Example

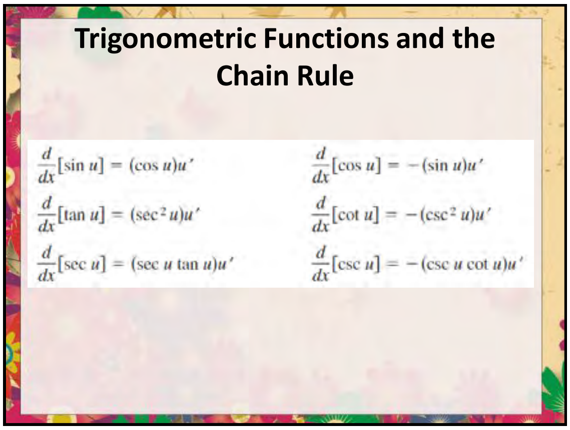
$$y = \left(\frac{3x-1}{x^2+3}\right)^2$$ Original function

$$y' = \overset{n}{2}\overbrace{\left(\frac{3x-1}{x^2+3}\right)}^{u^{n-1}}\overbrace{\frac{d}{dx}\left[\frac{3x-1}{x^2+3}\right]}^{u'}$$ General Power Rule

$$= \left[\frac{2(3x-1)}{x^2+3}\right]\left[\frac{(x^2+3)(3)-(3x-1)(2x)}{(x^2+3)^2}\right]$$ Quotient Rule

$$= \frac{2(3x-1)(3x^2+9-6x^2+2x)}{(x^2+3)^3}$$ Multiply.

$$= \frac{2(3x-1)(-3x^2+2x+9)}{(x^2+3)^3}$$ Simplify.

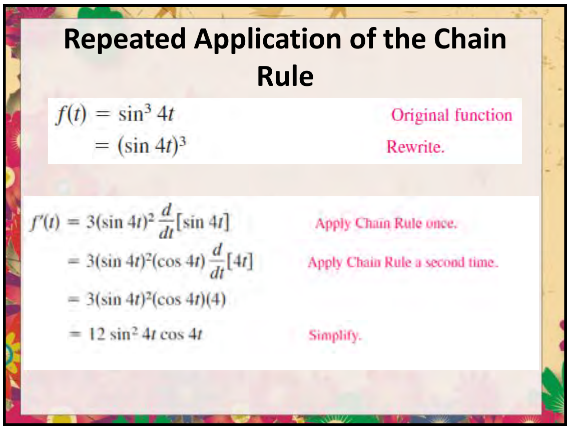


Trigonometric Functions and the Chain Rule

$\frac{d}{dx}[\sin u] = (\cos u)u'$ $\frac{d}{dx}[\cos u] = -(\sin u)u'$

$\frac{d}{dx}[\tan u] = (\sec^2 u)u'$ $\frac{d}{dx}[\cot u] = -(\csc^2 u)u'$

$\frac{d}{dx}[\sec u] = (\sec u \tan u)u'$ $\frac{d}{dx}[\csc u] = -(\csc u \cot u)u'$



Repeated Application of the Chain Rule

$f(t) = \sin^3 4t$ Original function

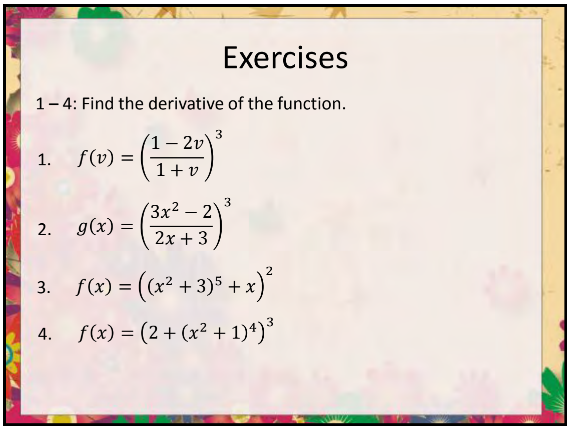
$= (\sin 4t)^3$ Rewrite.

$f'(t) = 3(\sin 4t)^2 \frac{d}{dt}[\sin 4t]$ Apply Chain Rule once.

$= 3(\sin 4t)^2(\cos 4t)\frac{d}{dt}[4t]$ Apply Chain Rule a second time.

$= 3(\sin 4t)^2(\cos 4t)(4)$

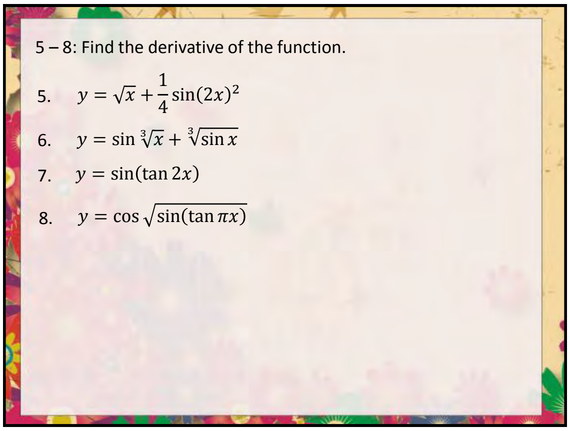
$= 12 \sin^2 4t \cos 4t$ Simplify.



Exercises

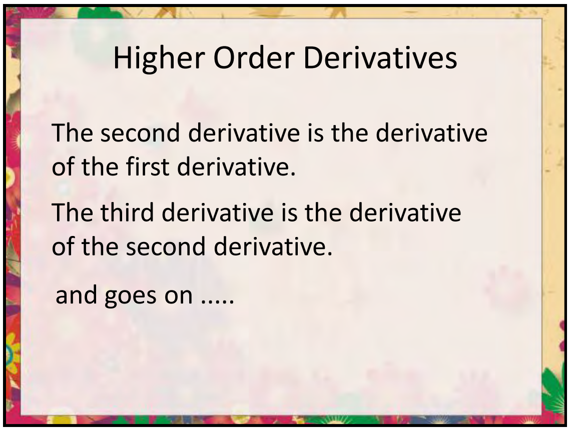
1 – 4: Find the derivative of the function.

1. $f(v) = \left(\frac{1-2v}{1+v}\right)^3$
2. $g(x) = \left(\frac{3x^2-2}{2x+3}\right)^3$
3. $f(x) = \left((x^2+3)^5 + x\right)^2$
4. $f(x) = \left(2 + (x^2+1)^4\right)^3$



5 – 8: Find the derivative of the function.

5. $y = \sqrt{x} + \frac{1}{4}\sin(2x)^2$
6. $y = \sin \sqrt[3]{x} + \sqrt[3]{\sin x}$
7. $y = \sin(\tan 2x)$
8. $y = \cos\sqrt{\sin(\tan \pi x)}$

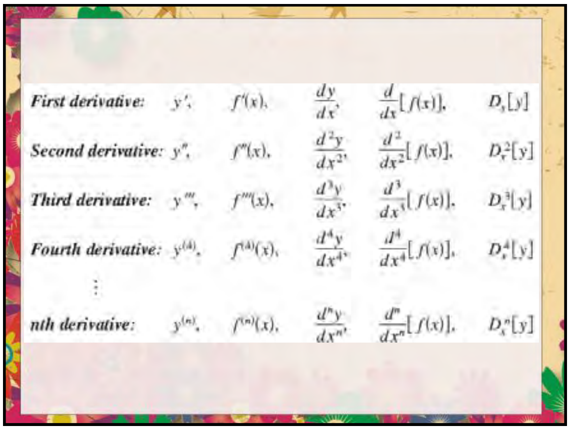


Higher Order Derivatives

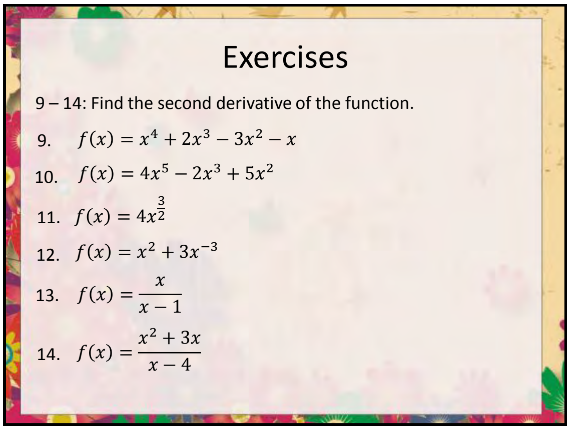
The second derivative is the derivative of the first derivative.

The third derivative is the derivative of the second derivative.

and goes on .....



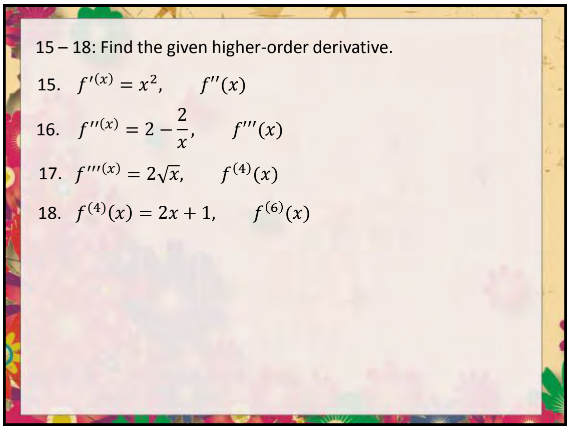
First derivative:	y',	$f'(x)$,	$\frac{dy}{dx}$,	$\frac{d}{dx}[f(x)]$,	$D_x[y]$
Second derivative:	y'',	$f''(x)$,	$\frac{d^2y}{dx^2}$,	$\frac{d^2}{dx^2}[f(x)]$,	$D_x^2[y]$
Third derivative:	y''',	$f'''(x)$,	$\frac{d^3y}{dx^3}$,	$\frac{d^3}{dx^3}[f(x)]$,	$D_x^3[y]$
Fourth derivative:	$y^{(4)}$,	$f^{(4)}(x)$,	$\frac{d^4y}{dx^4}$,	$\frac{d^4}{dx^4}[f(x)]$,	$D_x^4[y]$
$\vdots$					
nth derivative:	$y^{(n)}$,	$f^{(n)}(x)$,	$\frac{d^ny}{dx^n}$,	$\frac{d^n}{dx^n}[f(x)]$,	$D_x^n[y]$



Exercises

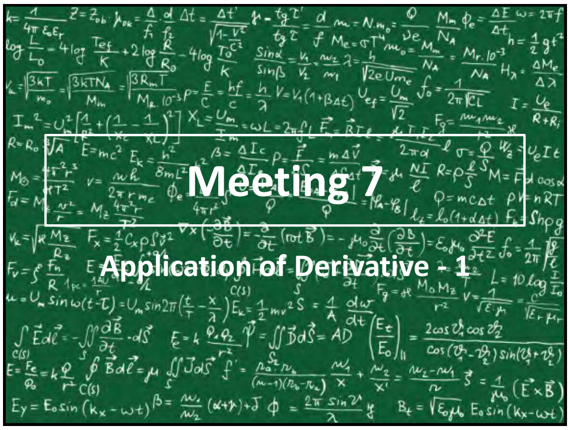
9 – 14: Find the second derivative of the function.

9. $f(x) = x^4 + 2x^3 - 3x^2 - x$
10. $f(x) = 4x^5 - 2x^3 + 5x^2$
11. $f(x) = 4x^{\frac{3}{2}}$
12. $f(x) = x^2 + 3x^{-3}$
13. $f(x) = \dfrac{x}{x-1}$
14. $f(x) = \dfrac{x^2+3x}{x-4}$



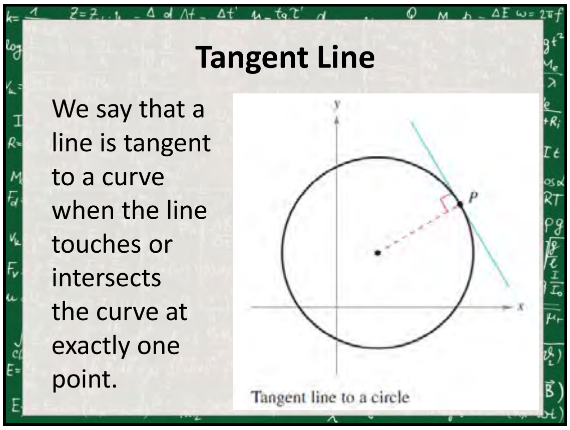
15 – 18: Find the given higher-order derivative.

15. $f'^{(x)} = x^2, \qquad f''(x)$
16. $f''^{(x)} = 2 - \frac{2}{x}, \qquad f'''(x)$
17. $f'''^{(x)} = 2\sqrt{x}, \qquad f^{(4)}(x)$
18. $f^{(4)}(x) = 2x + 1, \qquad f^{(6)}(x)$



Meeting 7

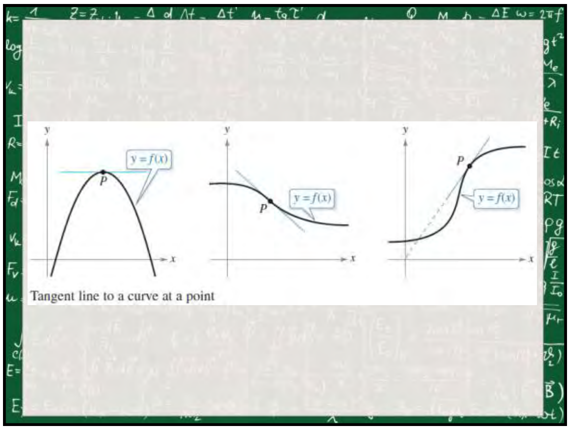
Application of Derivative - 1



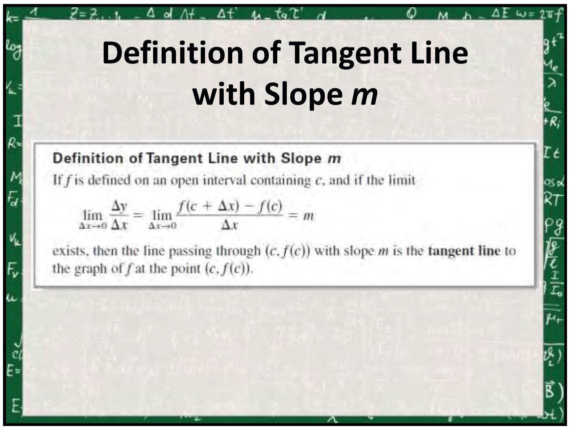
Tangent Line

We say that a line is tangent to a curve when the line touches or intersects the curve at exactly one point.

Tangent line to a circle



Tangent line to a curve at a point



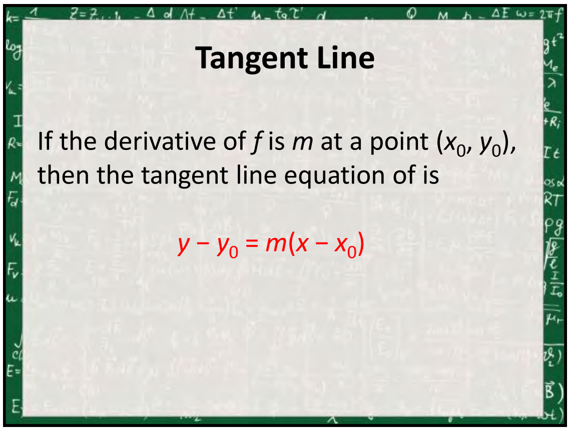
Definition of Tangent Line with Slope *m*

Definition of Tangent Line with Slope *m*

If f is defined on an open interval containing c, and if the limit

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = m$$

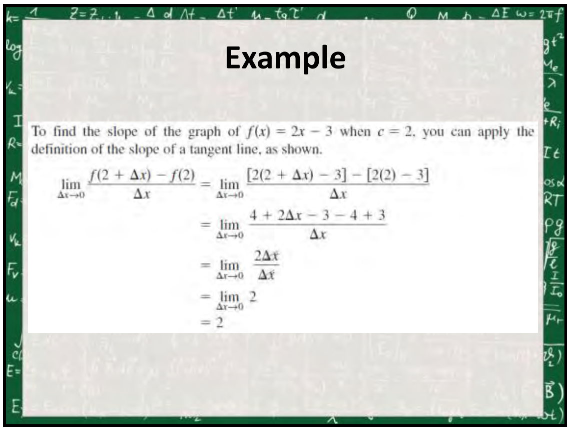
exists, then the line passing through $(c, f(c))$ with slope m is the **tangent line** to the graph of f at the point $(c, f(c))$.



Tangent Line

If the derivative of f is m at a point (x_0, y_0), then the tangent line equation of is

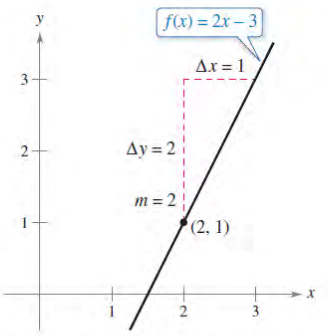
$$y - y_0 = m(x - x_0)$$



Example

To find the slope of the graph of $f(x) = 2x - 3$ when $c = 2$, you can apply the definition of the slope of a tangent line, as shown.

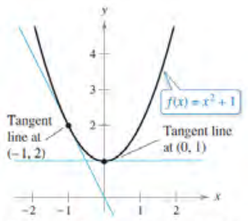
$$\begin{aligned}
\lim_{\Delta x \to 0} \frac{f(2 + \Delta x) - f(2)}{\Delta x} &= \lim_{\Delta x \to 0} \frac{[2(2 + \Delta x) - 3] - [2(2) - 3]}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{4 + 2\Delta x - 3 - 4 + 3}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{2\Delta x}{\Delta x} \\
&= \lim_{\Delta x \to 0} 2 \\
&= 2
\end{aligned}$$



The slope of f at (2, 1) is $m = 2$.

Tangent Lines to the Graph of a Nonlinear Function

Find the slopes of the tangent lines to the graph of $f(x) = x^2 + 1$ at the points (0, 1) and (−1, 2).



The slope of f at any point $(c, f(c))$ is $m = 2c$.

The slope is $m = 2(-1) = -2$

Normal Line

The *normal line* to a curve at one of its points (x_0, y_0) is the line that passes through the point and is perpendicular to the tangent line at that point.

If $m \neq 0$ is the slope of the tangent line, then the normal line equation is

$$y - y_0 = -(1/m)(x - x_0)$$

Example

Find equations of the tangent and normal lines to the parabola $y = 4x^2$ at the point (−1, 4).

Ans. $y + 8x + 4 = 0$; $8y - x - 33 = 0$

Exercise

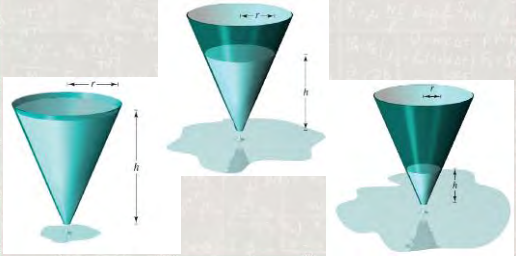
1. Find equations of the tangent and normal lines to $y = f(x) = x^3 - 2x^2 + 4$ at (2, 4).

Finding Related Rates

The important use of the Chain Rule is to find the rates of change of two or more related variables that are changing with respect to *time*.

For example, when water is drained out of a conical tank, the volume *V*, the radius *r*, and the height *h* of the water level are all functions of time.

$$V = \frac{\pi}{3}r^2h$$ Original equation



$$\frac{d}{dt}[V] = \frac{d}{dt}\left[\frac{\pi}{3}r^2h\right]$$

$$\frac{dV}{dt} = \frac{\pi}{3}\left[r^2\frac{dh}{dt} + h\left(2r\frac{dr}{dt}\right)\right]$$ Differentiate with respect to *t*.

$$= \frac{\pi}{3}\left(r^2\frac{dh}{dt} + 2rh\frac{dr}{dt}\right).$$

Example

The variables *x* and *y* are both differentiable functions of *t* and are related by the equation $y = x^2 + 3$. Find dy/dt when $x = 1$, given that $dx/dt = 2$ when $x = 1$.

Solution Using the Chain Rule, you can differentiate both sides of the equation *with respect to t*.

$$y = x^2 + 3$$ Write original equation.

$$\frac{d}{dt}[y] = \frac{d}{dt}[x^2 + 3]$$ Differentiate with respect to *t*.

$$\frac{dy}{dt} = 2x\frac{dx}{dt}$$ Chain Rule

When $x = 1$ and $dx/dt = 2$, you have

$$\frac{dy}{dt} = 2(1)(2) = 4.$$

Problem Solving with Related Rates

Based on previous example, we have

Equation: $y = x^2 + 3$

Given rate: $\frac{dx}{dt} = 2$ when $x = 1$

Find: $\frac{dy}{dt}$ when $x = 1$

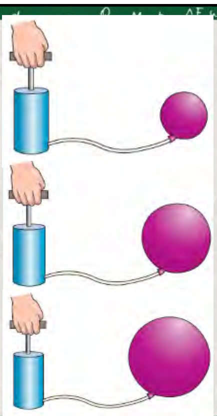
GUIDELINES FOR SOLVING RELATED-RATE PROBLEMS

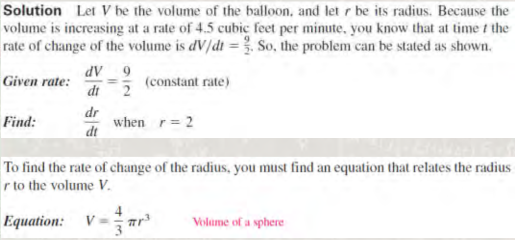
1. Identify all *given* quantities and quantities *to be determined*. Make a sketch and label the quantities.
2. Write an equation involving the variables whose rates of change either are given or are to be determined.
3. Using the Chain Rule, implicitly differentiate both sides of the equation *with respect to time t*.
4. *After* completing Step 3, substitute into the resulting equation all known values for the variables and their rates of change. Then solve for the required rate of change.

Verbal Statement	Mathematical Model
The velocity of a car after traveling for 1 hour is 50 miles per hour.	x = distance traveled $\frac{dx}{dt} = 50$ mi/h when $t = 1$
Water is being pumped into a swimming pool at a rate of 10 cubic meters per hour.	V = volume of water in pool $\frac{dV}{dt} = 10$ m^3/h
A gear is revolving at a rate of 25 revolutions per minute (1 revolution = 2π rad).	θ = angle of revolution $\frac{d\theta}{dt} = 25(2\pi)$ rad/min
A population of bacteria is increasing at a rate of 2000 per hour.	x = number in population $\frac{dx}{dt} = 2000$ bacteria per hour

Example

Air is being pumped into a spherical balloon (see below Figure) at a rate of 4.5 cubic feet per minute. Find the rate of change of the radius when the radius is 2 feet.





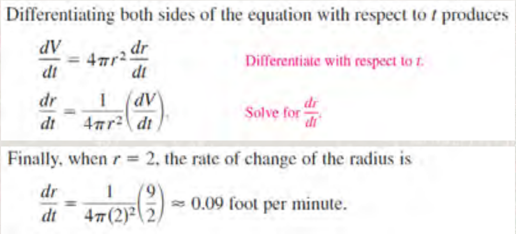
Solution Let V be the volume of the balloon, and let r be its radius. Because the volume is increasing at a rate of 4.5 cubic feet per minute, you know that at time t the rate of change of the volume is $dV/dt = \frac{9}{2}$. So, the problem can be stated as shown.

Given rate: $\frac{dV}{dt} = \frac{9}{2}$ (constant rate)

Find: $\frac{dr}{dt}$ when $r = 2$

To find the rate of change of the radius, you must find an equation that relates the radius r to the volume V.

Equation: $V = \frac{4}{3}\pi r^3$ Volume of a sphere



Differentiating both sides of the equation with respect to t produces

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$ Differentiate with respect to t.

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\left(\frac{dV}{dt}\right).$$ Solve for $\frac{dr}{dt}$.

Finally, when $r = 2$, the rate of change of the radius is

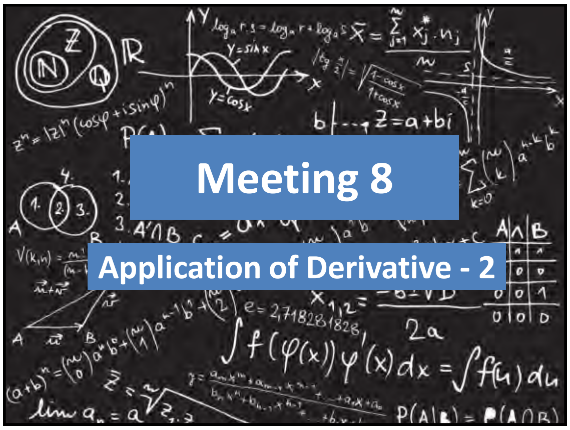
$$\frac{dr}{dt} = \frac{1}{4\pi(2)^2}\left(\frac{9}{2}\right) \approx 0.09 \text{ foot per minute.}$$

Exercises

2. The radius r of a circle is increasing at a rate of 4 centimeters per minute. Find the rates of changes of the area when
   (a). $r = 8$ centimeters
   (b). $r = 32$ centimeters

3. The radius r of a sphere is increasing at a rate of 4 inches per minute.
   (a). Find the rates of changes of the volume when $r = 9$ inches and $r = 36$ inches.
   (b). Explain why the rate of change of the volume of the sphere is not constant even though $\frac{dr}{dt}$ is constant.

4. A conical tank (with vertex down) is 10 feet across the top and 12 feet deep. Water is flowing into the tank at a rate of 10 cubic feet per minute. Find the rate of change of the depth of the water when the water is 8 feet deep.



Meeting 8

Application of Derivative - 2

Minimum/Maximum of A Function

Let f be defined on an interval I containing c.

1. $f(c)$ is the **minimum of f on I** when $f(c) \leq f(x)$ for all x in I.
2. $f(c)$ is the **maximum of f on I** when $f(c) \geq f(x)$ for all x in I.

The minimum and maximum of a function on an interval are the **extreme values,** or **extrema** (the singular form of extrema is extremum), of the function on the interval. The minimum and maximum of a function on an interval are also called the **absolute minimum** and **absolute maximum,** or the **global minimum** and **global maximum,** on the interval. Extrema that occur at the endpoints are called **endpoint extrema.**

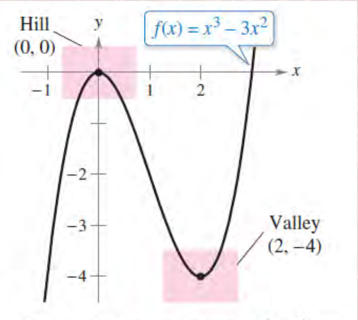
The Existence of the Extreme Value

THEOREM **The Extreme Value Theorem**

If f is continuous on a closed interval $[a, b]$, then f has both a minimum and a maximum on the interval.

Definition of Relative Extrema

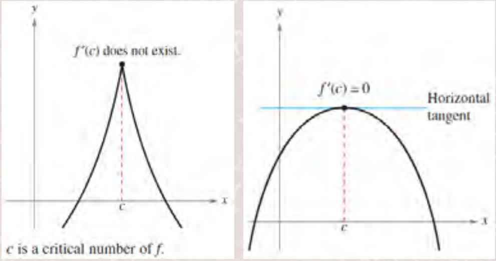
1. If there is an open interval containing c on which $f(c)$ is a maximum, then $f(c)$ is called a **relative maximum** of f, or you can say that f has a **relative maximum at $(c, f(c))$.**
2. If there is an open interval containing c on which $f(c)$ is a minimum, then $f(c)$ is called a **relative minimum** of f, or you can say that f has a **relative minimum at $(c, f(c))$.**



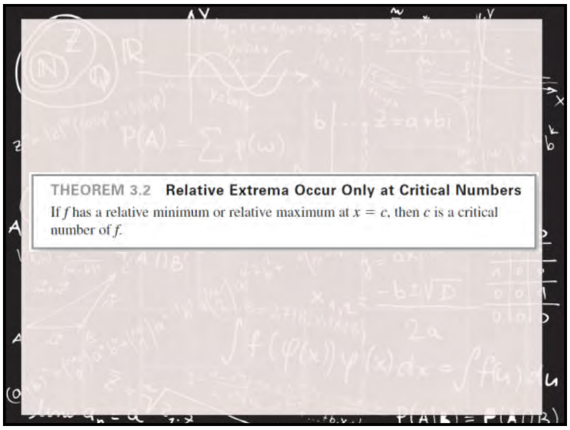
f has a relative maximum at (0, 0) and a relative minimum at (2, −4).

Definition of a Critical Number

Let f be defined at c. If $f'(c) = 0$ or if f is not differentiable at c, then c is a **critical number** of f.

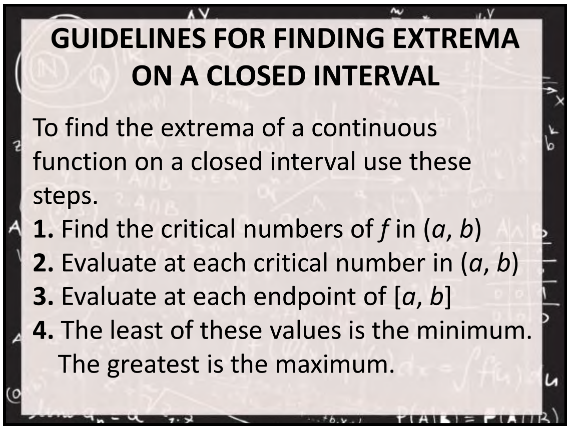


c is a critical number of f.



THEOREM 3.2 Relative Extrema Occur Only at Critical Numbers

If f has a relative minimum or relative maximum at $x = c$, then c is a critical number of f.



GUIDELINES FOR FINDING EXTREMA ON A CLOSED INTERVAL

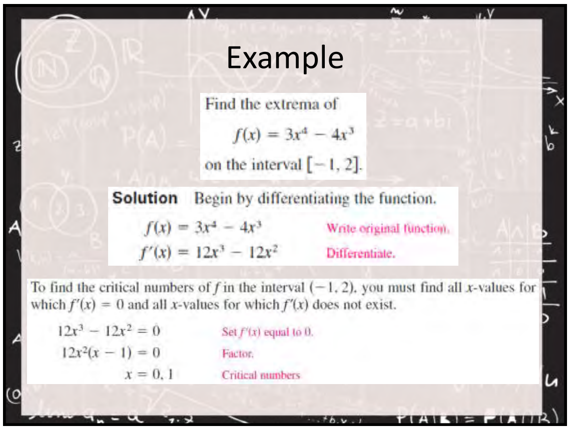
To find the extrema of a continuous function on a closed interval use these steps.

1. Find the critical numbers of f in (a, b)

2. Evaluate at each critical number in (a, b)

3. Evaluate at each endpoint of $[a, b]$

4. The least of these values is the minimum. The greatest is the maximum.



Example

Find the extrema of

$$f(x) = 3x^4 - 4x^3$$

on the interval $[-1, 2]$.

Solution Begin by differentiating the function.

$$f(x) = 3x^4 - 4x^3 \quad \text{Write original function.}$$

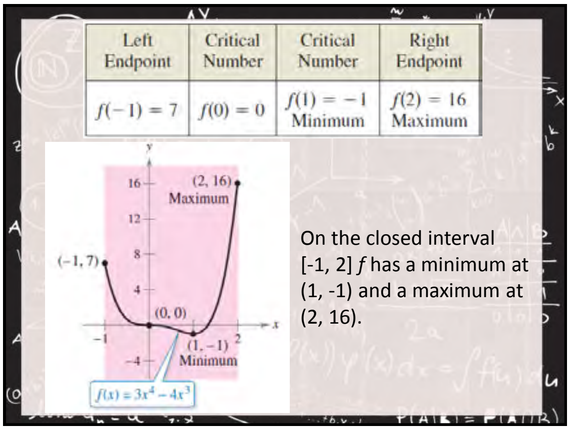
$$f'(x) = 12x^3 - 12x^2 \quad \text{Differentiate.}$$

To find the critical numbers of f in the interval $(-1, 2)$, you must find all x-values for which $f'(x) = 0$ and all x-values for which $f'(x)$ does not exist.

$$12x^3 - 12x^2 = 0 \quad \text{Set } f'(x) \text{ equal to 0.}$$

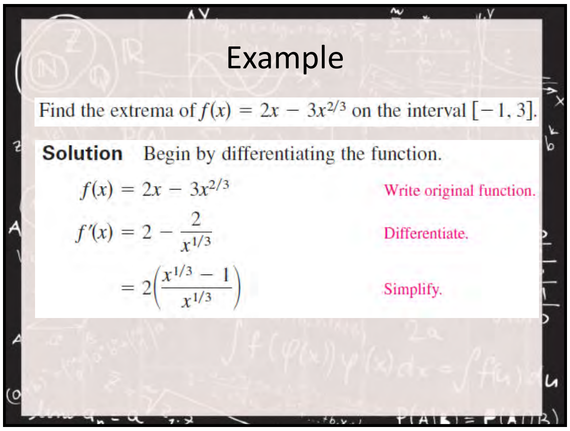
$$12x^2(x - 1) = 0 \quad \text{Factor.}$$

$$x = 0, 1 \quad \text{Critical numbers}$$



Left Endpoint	Critical Number	Critical Number	Right Endpoint
$f(-1) = 7$	$f(0) = 0$	$f(1) = -1$ Minimum	$f(2) = 16$ Maximum

On the closed interval [-1, 2] f has a minimum at (1, -1) and a maximum at (2, 16).



Example

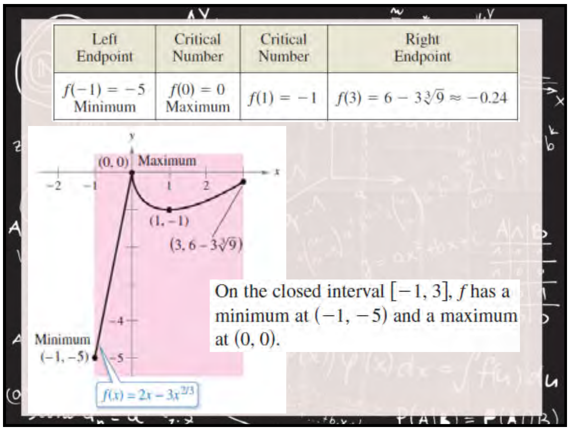
Find the extrema of $f(x) = 2x - 3x^{2/3}$ on the interval $[-1, 3]$.

Solution Begin by differentiating the function.

$$f(x) = 2x - 3x^{2/3} \quad \text{Write original function.}$$

$$f'(x) = 2 - \frac{2}{x^{1/3}} \quad \text{Differentiate.}$$

$$= 2\left(\frac{x^{1/3} - 1}{x^{1/3}}\right) \quad \text{Simplify.}$$



Left Endpoint	Critical Number	Critical Number	Right Endpoint
$f(-1) = -5$ Minimum	$f(0) = 0$ Maximum	$f(1) = -1$	$f(3) = 6 - 3\sqrt[3]{9} \approx -0.24$

On the closed interval $[-1, 3]$, f has a minimum at $(-1, -5)$ and a maximum at $(0, 0)$.

Exercises

1 – 4: Find the critical numbers of the function.

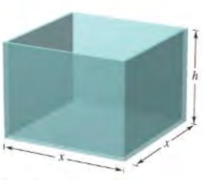
1. $f(x) = x^3 - 3x^2$
2. $g(x) = x^4 - 8x^2$
3. $g(t) = t\sqrt{4-t}, t < 3$
4. $f(x) = \dfrac{4x}{x^2+1}$

5 – 6: Find the absolute extrema of the function on the closed interval.

5. $g(t) = \dfrac{t^2}{t^2+3}, \quad [-1, 1]$
6. $f(t) = \dfrac{2x}{x^2+1}, \quad [-2, 2]$

Applied Minimum and Maximum Problems

A manufacturer wants to design an open box having a square base and a surface area of 108 square inches, as shown in Figure 3.53. What dimensions will produce a box with maximum volume?



Open box with square base:
$S = x^2 + 4xh = 108$

Solution Because the box has a square base, its volume is

$$V = x^2h. \qquad \text{Primary equation}$$

This equation is called the **primary equation** because it gives a formula for the quantity to be optimized. The surface area of the box is

$$S = (\text{area of base}) + (\text{area of four sides})$$
$$108 = x^2 + 4xh. \qquad \text{Secondary equation}$$

$$0 \le x \le \sqrt{108}. \qquad \text{Feasible domain}$$

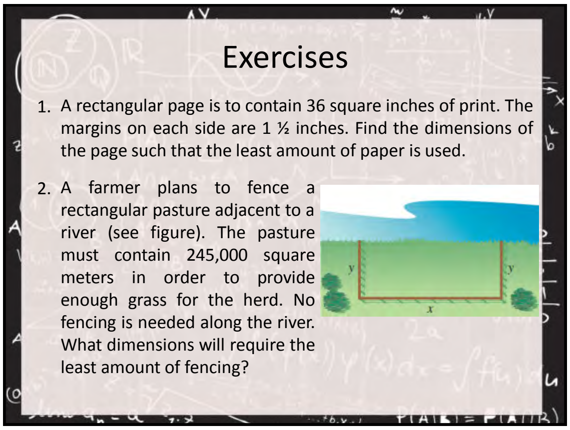
$$V = x^2h \qquad \text{Function of two variables}$$
$$= x^2\left(\frac{108 - x^2}{4x}\right) \qquad \text{Substitute for } h.$$
$$= 27x - \frac{x^3}{4}. \qquad \text{Function of one variable}$$

$$\frac{dV}{dx} = 27 - \frac{3x^2}{4} \qquad \text{Differentiate with respect to } x.$$
$$27 - \frac{3x^2}{4} = 0 \qquad \text{Set derivative equal to 0.}$$
$$3x^2 = 108 \qquad \text{Simplify.}$$
$$x = \pm 6 \qquad \text{Critical numbers}$$

So, the critical numbers are $x = \pm 6$. You do not need to consider $x = -6$ because it is outside the domain. Evaluating V at the critical number $x = 6$ and at the endpoints of the domain produces $V(0) = 0$, $V(6) = 108$, and $V(\sqrt{108}) = 0$. So, V is maximum when $x = 6$, and the dimensions of the box are 6 inches by 6 inches by 3 inches. ■

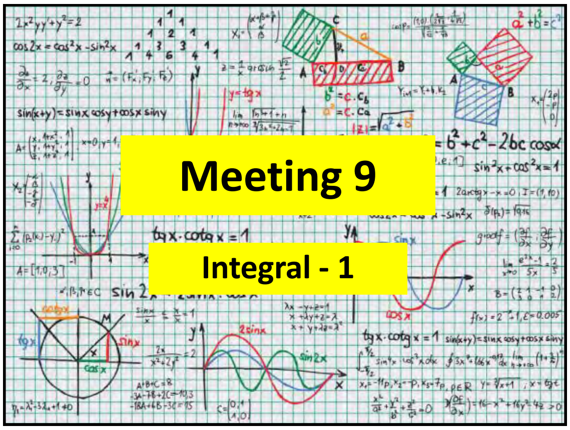
GUIDELINES FOR SOLVING APPLIED MINIMUM AND MAXIMUM PROBLEMS

1. Identify all *given* quantities and all quantities *to be determined*. If possible, make a sketch.
2. Write a **primary equation** for the quantity that is to be maximized or minimized. (A review of several useful formulas from geometry is presented inside the back cover.)
3. Reduce the primary equation to one having a *single independent variable*. This may involve the use of **secondary equations** relating the independent variables of the primary equation.
4. Determine the feasible domain of the primary equation. That is, determine the values for which the stated problem makes sense.
5. Determine the desired maximum or minimum value by the calculus techniques.



Exercises

1. A rectangular page is to contain 36 square inches of print. The margins on each side are 1 ½ inches. Find the dimensions of the page such that the least amount of paper is used.
2. A farmer plans to fence a rectangular pasture adjacent to a river (see figure). The pasture must contain 245,000 square meters in order to provide enough grass for the herd. No fencing is needed along the river. What dimensions will require the least amount of fencing?



Meeting 9

Integral - 1

Antiderivatives

To find a function F whose derivative is $3x^2$, we might use our knowledge of derivatives to conclude that

$$F(x) = x^3 \text{ because } \frac{d[x^3]}{dx} = 3x^2.$$

The function F is an *antiderivative* of f.

Definition of Antiderivative

A function F is an **antiderivative** of f on an interval I when

$$F'(x) = f(x)$$

for all x in I.

Example

Find the general solution of the differential equation $y' = 2$.

Solution To begin, you need to find a function whose derivative is 2. One such function is

$y = 2x.$ $2x$ is *an* antiderivative of 2.

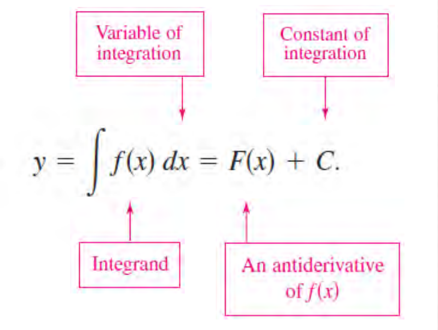
$y = 2x + C.$ General solution

Antidifferentiation

Consider to solving this following differential equation

$$\frac{dy}{dx} = f(x)$$

The operation of finding all solutions of this equation is called **antidifferentiation** (or **indefinite integration**) and is denoted by an integral sign $\int$.



Variable of integration

Constant of integration

$$y = \int f(x)\,dx = F(x) + C.$$

Integrand

An antiderivative of $f(x)$

The expression $\int f(x)dx$ is read as the *antiderivative of f with respect to x*. So, the differential dx serves to identify x as the variable of integration.

The term **Indefinite integral** is a synonym for antiderivative.

Basic Integration Rules

$$\int F'(x)\,dx = F(x) + C.$$

$$\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x).$$

Basic Integration Rules

Differentiation Formula	Integration Formula	
$\frac{d}{dx}[C] = 0$	$\int 0\,dx = C$	
$\frac{d}{dx}[kx] = k$	$\int k\,dx = kx + C$	
$\frac{d}{dx}[kf(x)] = kf'(x)$	$\int kf(x)\,dx = k\int f(x)\,dx$	
$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$	$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$	
$\frac{d}{dx}[x^n] = nx^{n-1}$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\quad n \neq -1$	Power Rule

Basic Integration Rules

$\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x\,dx = \sin x + C$
$\frac{d}{dx}[\cos x] = -\sin x$	$\int \sin x\,dx = -\cos x + C$
$\frac{d}{dx}[\tan x] = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$
$\frac{d}{dx}[\sec x] = \sec x \tan x$	$\int \sec x \tan x\,dx = \sec x + C$
$\frac{d}{dx}[\cot x] = -\csc^2 x$	$\int \csc^2 x\,dx = -\cot x + C$
$\frac{d}{dx}[\csc x] = -\csc x \cot x$	$\int \csc x \cot x\,dx = -\csc x + C$

The general pattern of integration

Original integral

⇩

Rewrite

⇩

Integrate

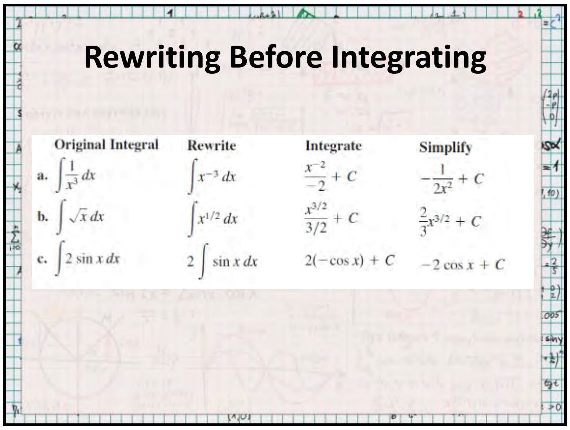
⇩

Simplify

Example

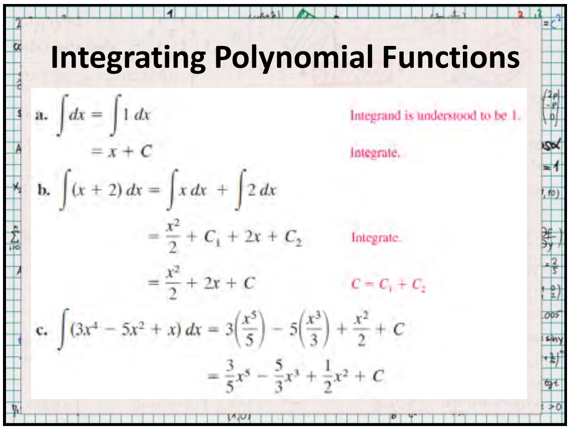
Integrate

$\int 3x\,dx = 3\int x\,dx$	Constant Multiple Rule
$= 3\int x^1\,dx$	Rewrite x as x^1.
$= 3\left(\frac{x^2}{2}\right) + C$	Power Rule ($n = 1$)
$= \frac{3}{2}x^2 + C$	Simplify.



Rewriting Before Integrating

	Original Integral	Rewrite	Integrate	Simplify
a.	$\int \frac{1}{x^3}\,dx$	$\int x^{-3}\,dx$	$\frac{x^{-2}}{-2} + C$	$-\frac{1}{2x^2} + C$
b.	$\int \sqrt{x}\,dx$	$\int x^{1/2}\,dx$	$\frac{x^{3/2}}{3/2} + C$	$\frac{2}{3}x^{3/2} + C$
c.	$\int 2 \sin x\,dx$	$2\int \sin x\,dx$	$2(-\cos x) + C$	$-2 \cos x + C$



Integrating Polynomial Functions

a. $\int dx = \int 1\,dx$ Integrand is understood to be 1.

$= x + C$ Integrate.

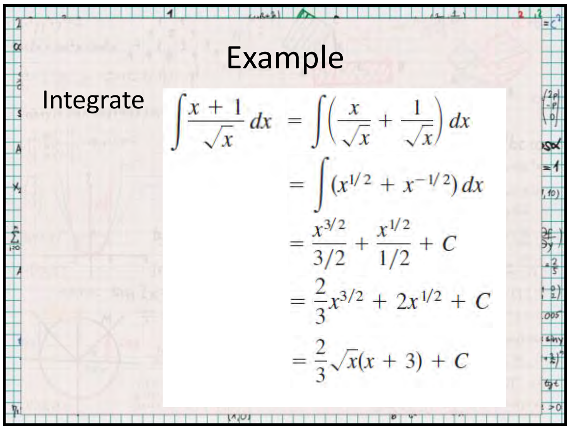
b. $\int (x + 2)\,dx = \int x\,dx + \int 2\,dx$

$= \frac{x^2}{2} + C_1 + 2x + C_2$ Integrate.

$= \frac{x^2}{2} + 2x + C$ $C = C_1 + C_2$

c. $\int (3x^4 - 5x^2 + x)\,dx = 3\left(\frac{x^5}{5}\right) - 5\left(\frac{x^3}{3}\right) + \frac{x^2}{2} + C$

$= \frac{3}{5}x^5 - \frac{5}{3}x^3 + \frac{1}{2}x^2 + C$



Example

Integrate

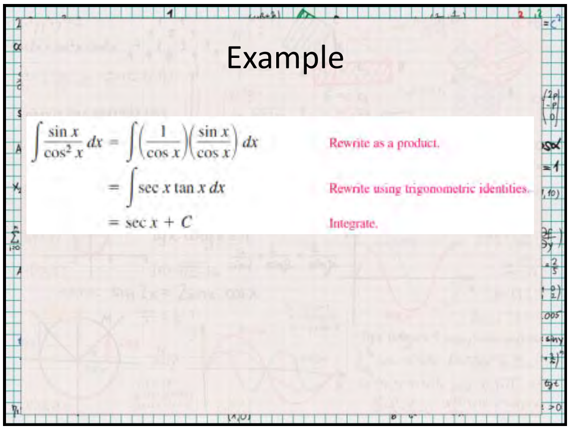
$$\int \frac{x+1}{\sqrt{x}}\,dx = \int \left(\frac{x}{\sqrt{x}} + \frac{1}{\sqrt{x}}\right) dx$$

$$= \int (x^{1/2} + x^{-1/2})\,dx$$

$$= \frac{x^{3/2}}{3/2} + \frac{x^{1/2}}{1/2} + C$$

$$= \frac{2}{3}x^{3/2} + 2x^{1/2} + C$$

$$= \frac{2}{3}\sqrt{x}(x + 3) + C$$

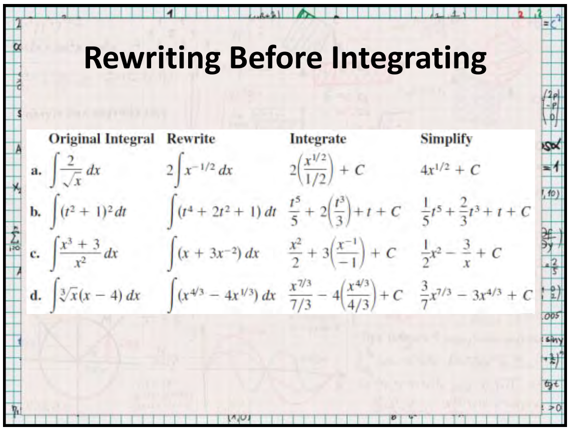


Example

$\int \frac{\sin x}{\cos^2 x}\,dx = \int \left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{\cos x}\right) dx$ Rewrite as a product.

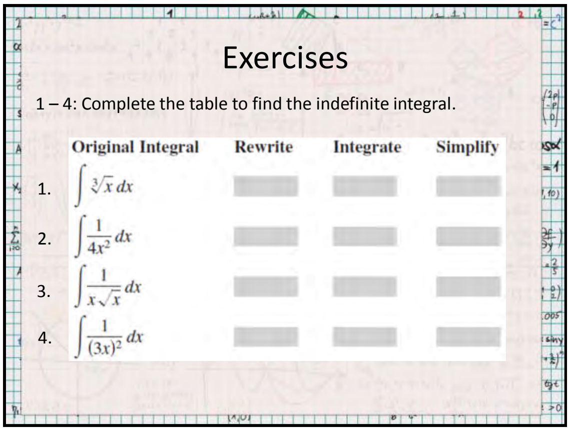
$= \int \sec x \tan x\,dx$ Rewrite using trigonometric identities.

$= \sec x + C$ Integrate.



Rewriting Before Integrating

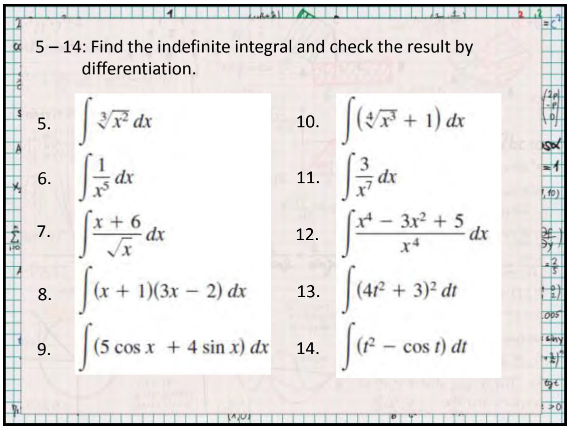
	Original Integral	Rewrite	Integrate	Simplify
a.	$\int \frac{2}{\sqrt{x}}\,dx$	$2\int x^{-1/2}\,dx$	$2\left(\frac{x^{1/2}}{1/2}\right) + C$	$4x^{1/2} + C$
b.	$\int (t^2 + 1)^2\,dt$	$\int (t^4 + 2t^2 + 1)\,dt$	$\frac{t^5}{5} + 2\left(\frac{t^3}{3}\right) + t + C$	$\frac{1}{5}t^5 + \frac{2}{3}t^3 + t + C$
c.	$\int \frac{x^3 + 3}{x^2}\,dx$	$\int (x + 3x^{-2})\,dx$	$\frac{x^2}{2} + 3\left(\frac{x^{-1}}{-1}\right) + C$	$\frac{1}{2}x^2 - \frac{3}{x} + C$
d.	$\int \sqrt[3]{x}(x - 4)\,dx$	$\int (x^{4/3} - 4x^{1/3})\,dx$	$\frac{x^{7/3}}{7/3} - 4\left(\frac{x^{4/3}}{4/3}\right) + C$	$\frac{3}{7}x^{7/3} - 3x^{4/3} + C$



Exercises

1 – 4: Complete the table to find the indefinite integral.

	Original Integral	Rewrite	Integrate	Simplify
1.	$\int \sqrt[3]{x}\,dx$			
2.	$\int \frac{1}{4x^2}\,dx$			
3.	$\int \frac{1}{x\sqrt{x}}\,dx$			
4.	$\int \frac{1}{(3x)^2}\,dx$			



5 – 14: Find the indefinite integral and check the result by differentiation.

5. $\int \sqrt[3]{x^2}\, dx$

6. $\int \frac{1}{x^5}\, dx$

7. $\int \frac{x + 6}{\sqrt{x}}\, dx$

8. $\int (x + 1)(3x - 2)\, dx$

9. $\int (5 \cos x\ + 4 \sin x)\, dx$

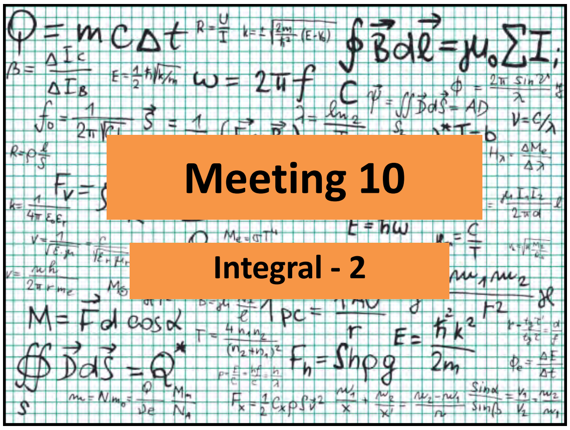
10. $\int \left(\sqrt[4]{x^3} + 1\right) dx$

11. $\int \frac{3}{x^7}\, dx$

12. $\int \frac{x^4 - 3x^2 + 5}{x^4}\, dx$

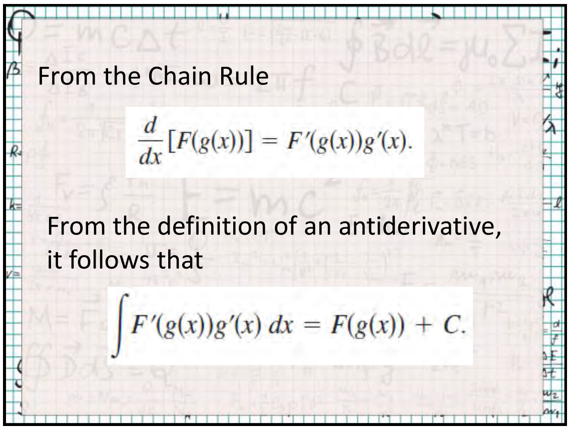
13. $\int (4t^2 + 3)^2\, dt$

14. $\int (t^2 - \cos t)\, dt$



Meeting 10

Integral - 2

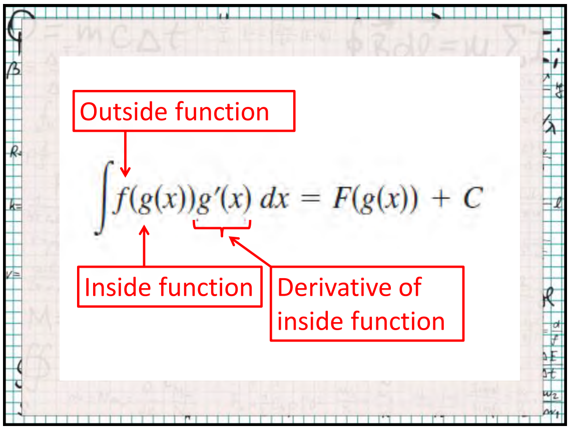


From the Chain Rule

$$\frac{d}{dx}[F(g(x))] = F'(g(x))g'(x).$$

From the definition of an antiderivative, it follows that

$$\int F'(g(x))g'(x)\,dx = F(g(x)) + C.$$

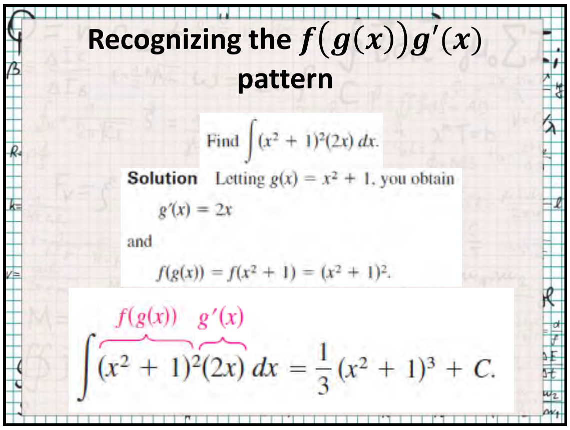


Outside function

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C$$

Inside function

Derivative of inside function



Recognizing the $f(g(x))g'(x)$ pattern

Find $\int (x^2 + 1)^2(2x)\,dx$.

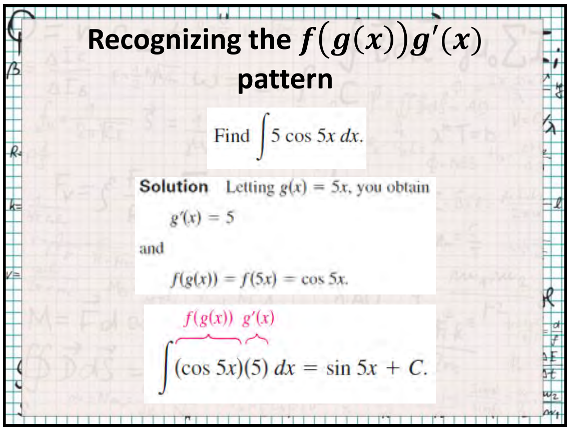
Solution Letting $g(x) = x^2 + 1$, you obtain

$$g'(x) = 2x$$

and

$$f(g(x)) = f(x^2 + 1) = (x^2 + 1)^2.$$

$$\int \overbrace{(x^2 + 1)^2}^{f(g(x))}\overbrace{(2x)}^{g'(x)}\,dx = \frac{1}{3}(x^2 + 1)^3 + C.$$



Recognizing the $f(g(x))g'(x)$ pattern

Find $\int 5\cos 5x\,dx$.

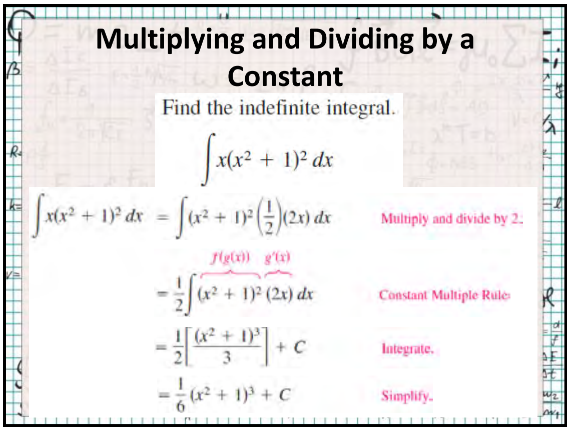
Solution Letting $g(x) = 5x$, you obtain

$$g'(x) = 5$$

and

$$f(g(x)) = f(5x) = \cos 5x.$$

$$\int \overbrace{(\cos 5x)}^{f(g(x))}\overbrace{(5)}^{g'(x)}\,dx = \sin 5x + C.$$



Multiplying and Dividing by a Constant

Find the indefinite integral.

$$\int x(x^2 + 1)^2\,dx$$

$$\int x(x^2 + 1)^2\,dx = \int (x^2 + 1)^2\left(\frac{1}{2}\right)(2x)\,dx$$ Multiply and divide by 2.

$$= \frac{1}{2}\int \overbrace{(x^2 + 1)^2}^{f(g(x))}\,\overbrace{(2x)}^{g'(x)}\,dx$$ Constant Multiple Rule.

$$= \frac{1}{2}\left[\frac{(x^2 + 1)^3}{3}\right] + C$$ Integrate.

$$= \frac{1}{6}(x^2 + 1)^3 + C$$ Simplify.

Change of Variables

The change of variables technique uses the Leibniz notation for the differential. That is, if $u = g(x)$, then $du = g'(x)$, and

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du = F(u) + C.$$

Example

Find $\int \sqrt{2x - 1}\,dx$.

$$\begin{aligned}\int \sqrt{2x-1}\,dx &= \int \sqrt{u}\left(\frac{du}{2}\right)\\ &= \frac{1}{2}\int u^{1/2}\,du\\ &= \frac{1}{2}\left(\frac{u^{3/2}}{3/2}\right) + C\\ &= \frac{1}{3}u^{3/2} + C\\ &= \frac{1}{3}(2x-1)^{3/2} + C.\end{aligned}$$

Example

Find $\int x\sqrt{2x - 1}\,dx$.

Solution As in the previous example, let $u = 2x - 1$ and obtain $dx = du/2$. Because the integrand contains a factor of x, you must also solve for x in terms of u, as shown.

$u = 2x - 1 \implies x = \frac{u+1}{2}$ Solve for x in terms of u.

$$\begin{aligned}\int x\sqrt{2x-1}\,dx &= \int \left(\frac{u+1}{2}\right)u^{1/2}\left(\frac{du}{2}\right)\\ &= \frac{1}{4}\int (u^{3/2} + u^{1/2})\,du\\ &= \frac{1}{4}\left(\frac{u^{5/2}}{5/2} + \frac{u^{3/2}}{3/2}\right) + C\\ &= \frac{1}{10}(2x-1)^{5/2} + \frac{1}{6}(2x-1)^{3/2} + C.\end{aligned}$$

GUIDELINES FOR MAKING A CHANGE OF VARIABLES

1. Choose a substitution $u = g(x)$. Usually, it is best to choose the *inner* part of a composite function, such as a quantity raised to a power.
2. Compute $du = g'(x)\,dx$.
3. Rewrite the integral in terms of the variable u.
4. Find the resulting integral in terms of u.
5. Replace u by $g(x)$ to obtain an antiderivative in terms of x.
6. Check your answers by differentiating.

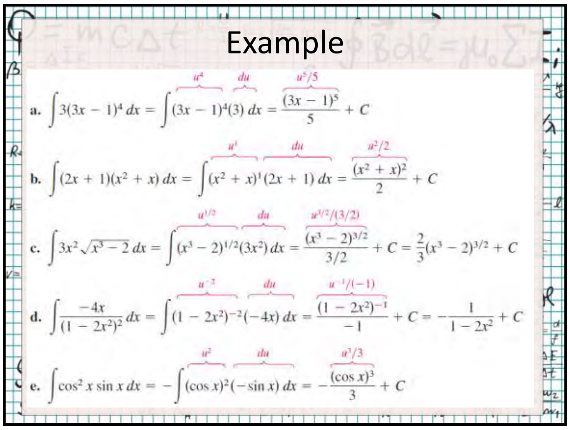
The General Power Rule for Integration

If g is a differentiable function of x, then

$$\int [g(x)]^n g'(x)\,dx = \frac{[g(x)]^{n+1}}{n+1} + C, \quad n \neq -1.$$

Equivalently, if $u = g(x)$, then

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1.$$



Example

a. $\int 3(3x-1)^4\,dx = \int \overbrace{(3x-1)^4}^{u^4}\overbrace{(3)\,dx}^{du} = \overbrace{\frac{(3x-1)^5}{5}}^{u^5/5} + C$

b. $\int (2x+1)(x^2+x)\,dx = \int \overbrace{(x^2+x)^1}^{u^1}\overbrace{(2x+1)\,dx}^{du} = \overbrace{\frac{(x^2+x)^2}{2}}^{u^2/2} + C$

c. $\int 3x^2\sqrt{x^3-2}\,dx = \int \overbrace{(x^3-2)^{1/2}}^{u^{1/2}}\overbrace{(3x^2)\,dx}^{du} = \overbrace{\frac{(x^3-2)^{3/2}}{3/2}}^{u^{3/2}/(3/2)} + C = \frac{2}{3}(x^3-2)^{3/2} + C$

d. $\int \frac{-4x}{(1-2x^2)^2}\,dx = \int \overbrace{(1-2x^2)^{-2}}^{u^{-2}}\overbrace{(-4x)\,dx}^{du} = \overbrace{\frac{(1-2x^2)^{-1}}{-1}}^{u^{-1}/(-1)} + C = -\frac{1}{1-2x^2} + C$

e. $\int \cos^2 x \sin x\,dx = -\int \overbrace{(\cos x)^2}^{u^2}\overbrace{(-\sin x)\,dx}^{du} = -\overbrace{\frac{(\cos x)^3}{3}}^{u^3/3} + C$

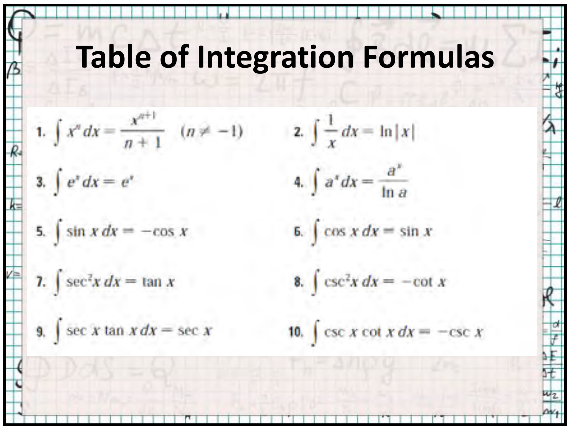


Table of Integration Formulas

1. $\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1)$
2. $\int \frac{1}{x}\,dx = \ln|x|$
3. $\int e^x\,dx = e^x$
4. $\int a^x\,dx = \frac{a^x}{\ln a}$
5. $\int \sin x\,dx = -\cos x$
6. $\int \cos x\,dx = \sin x$
7. $\int \sec^2 x\,dx = \tan x$
8. $\int \csc^2 x\,dx = -\cot x$
9. $\int \sec x \tan x\,dx = \sec x$
10. $\int \csc x \cot x\,dx = -\csc x$

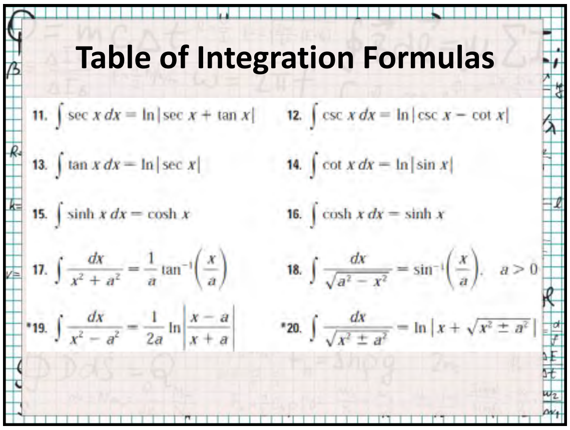
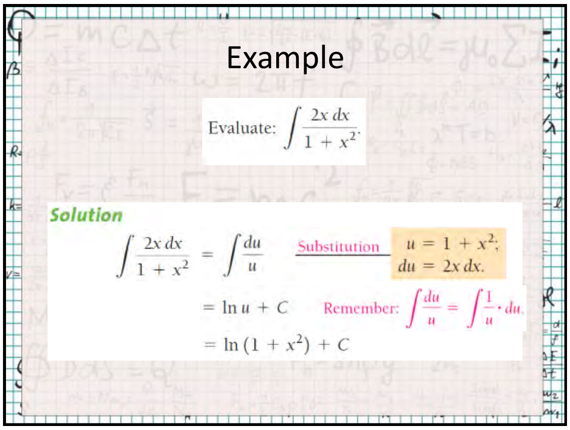


Table of Integration Formulas

11. $\int \sec x\,dx = \ln|\sec x + \tan x|$
12. $\int \csc x\,dx = \ln|\csc x - \cot x|$
13. $\int \tan x\,dx = \ln|\sec x|$
14. $\int \cot x\,dx = \ln|\sin x|$
15. $\int \sinh x\,dx = \cosh x$
16. $\int \cosh x\,dx = \sinh x$
17. $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$
18. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right), \quad a > 0$

*19. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$

*20. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right|$



Example

Evaluate: $\int \frac{2x\,dx}{1+x^2}$.

Solution

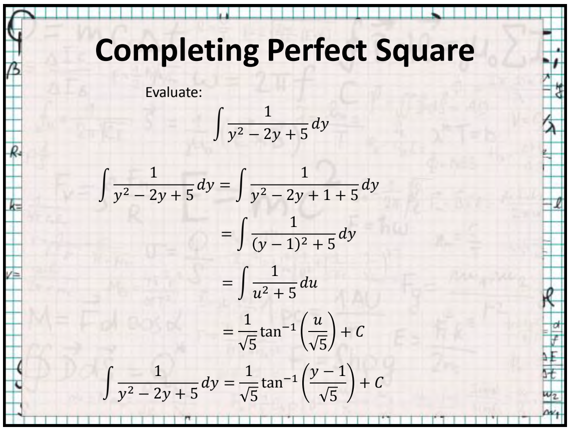
$$\int \frac{2x\,dx}{1+x^2} = \int \frac{du}{u}$$

Substitution $u = 1 + x^2;$ $du = 2x\,dx.$

$$= \ln u + C$$

Remember: $\int \frac{du}{u} = \int \frac{1}{u} \cdot du.$

$$= \ln(1+x^2) + C$$



Completing Perfect Square

Evaluate:

$$\int \frac{1}{y^2-2y+5}\,dy$$

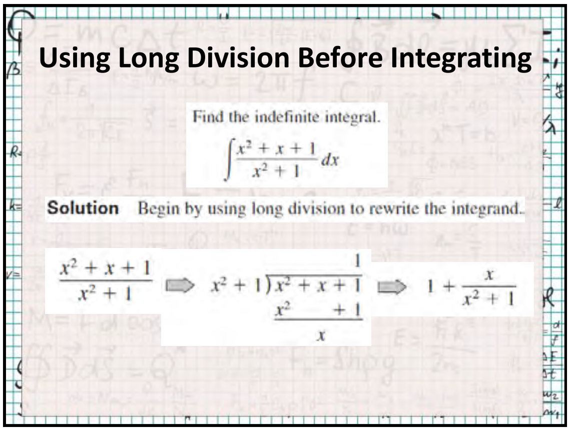
$$\int \frac{1}{y^2-2y+5}\,dy = \int \frac{1}{y^2-2y+1+5}\,dy$$

$$= \int \frac{1}{(y-1)^2+5}\,dy$$

$$= \int \frac{1}{u^2+5}\,du$$

$$= \frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{u}{\sqrt{5}}\right) + C$$

$$\int \frac{1}{y^2-2y+5}\,dy = \frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{y-1}{\sqrt{5}}\right) + C$$



Using Long Division Before Integrating

Find the indefinite integral.

$$\int \frac{x^2+x+1}{x^2+1}\,dx$$

Solution Begin by using long division to rewrite the integrand.

$$\frac{x^2+x+1}{x^2+1} \Rightarrow x^2+1\overline{\smash{)}\,x^2+x+1} \Rightarrow 1 + \frac{x}{x^2+1}$$

$$\begin{array}{r} 1 \\ x^2+1\overline{)\,x^2+x+1} \\ \underline{x^2 +1} \\ x \end{array}$$

$$\begin{aligned}\int \frac{x^2+x+1}{x^2+1}\,dx &= \int\left(1+\frac{x}{x^2+1}\right)dx\\ &= \int dx + \frac{1}{2}\int\frac{2x}{x^2+1}\,dx\\ &= x+\frac{1}{2}\ln(x^2+1)+C.\end{aligned}$$

Multiplying numerator and denominator by the same factor

Evaluate: $\int \sqrt{\frac{1-x}{1+x}}\,dx$

Solution: Multiplying numerator and denominator by $\sqrt{1-x}$

$$\begin{aligned}\int \sqrt{\frac{1-x}{1+x}}\,dx &= \int\frac{1-x}{\sqrt{1-x^2}}\,dx\\ &= \int\frac{1}{\sqrt{1-x^2}}\,dx - \int\frac{x}{\sqrt{1-x^2}}\,dx\\ &= \sin^{-1}x+\sqrt{1-x^2}+C\end{aligned}$$

Using Identity Trigonometry

Evaluate: $\int \frac{\tan^3 x}{\cos^3 x}\,dx$

Solution:

$$\int \frac{\tan^3 x}{\cos^3 x}\,dx = \int \frac{\sin^3 x}{\cos^3 x}\,\frac{1}{\cos^3 x}\,dx = \int\frac{\sin^3 x}{\cos^6 x}\,dx$$

$$\begin{aligned}\int \frac{\tan^3 x}{\cos^3 x}\,dx &= \int\frac{\sin^3 x}{\cos^6 x}\,dx\\ &= \int\frac{1-\cos^2 x}{\cos^6 x}\sin x\,dx\\ &= \int\frac{1-u^2}{u^6}(-du)\\ &= \int\frac{u^2-1}{u^6}\,du\\ &= \int(u^{-4}-u^{-6})du\\ &= -\frac{1}{3}u^{-3}+\frac{1}{5}u^{-5}+C\\ &= -\frac{1}{3}(\cos x)^{-3}+\frac{1}{5}(\cos x)^{-5}+C\end{aligned}$$

Exercises

1 – 2: Evaluate the integral.

1. $\int \frac{x^3+4}{x^2+4}\,dx$
2. $\int \frac{x+4}{x^2+2x+5}\,dx$

3 – 9: Find the indefinite integral by the method shown in the previous examples..

3. $\int x\sqrt{x+6}\,dx$
4. $\int x\sqrt{3x-4}\,dx$

---

5. $\int x^2\sqrt{1-x}\,dx$
6. $\int \frac{x^2-1}{\sqrt{2x-1}}\,dx$
7. $\int \frac{-x}{(x+1)-\sqrt{x+1}}\,dx$
8. $\int (x+1)\sqrt{2-x}\,dx$
9. $\int \frac{2x+1}{\sqrt{x+4}}\,dx$

Meeting 11

Integral - 3

Integration of Rational Functions by Partial Fractions

Integrate:

$$\int \frac{5x-3}{x^2-2x-3}dx$$

For instance, the rational function $\frac{5x-3}{x^2-2x-3}$ can be rewritten as

$$\frac{5x-3}{x^2-2x-3} = \frac{2}{x+1} + \frac{3}{x-3}.$$

Then, we have

$$\int \frac{5x-3}{(x+1)(x-3)}dx = \int \frac{2}{x+1}dx + \int \frac{3}{x-3}dx$$
$$= 2\ln|x+1| + 3\ln|x-3| + C.$$

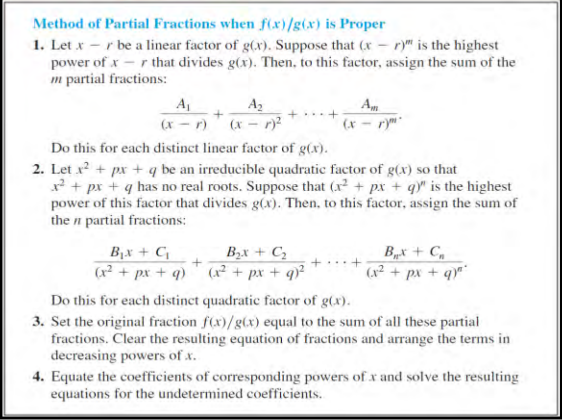
The method for rewriting rational functions as a sum of simpler fractions is called **the method of partial fractions**.

General Description of the Method

Success in writing a rational function *f*(*x*)/*g*(*x*) as a sum of partial fractions depends on two things:

- *The degree of f*(*x*) *must be less than the degree of g*(*x*).
  That is, the fraction must be proper.
  If it isn't, divide *f*(*x*) by *g*(*x*) and work with the remainder term.

- *We must know the factors of g*(*x*).
  In theory, any polynomial with real coefficients can be written as a product of real linear factors and real quadratic factors. In practice, the factors may be hard to find. A quadratic polynomial (or factor) is **irreducible** if it cannot be written as the product of two linear factors with real coefficients.



Method of Partial Fractions when $f(x)/g(x)$ is Proper

1. Let $x - r$ be a linear factor of $g(x)$. Suppose that $(x-r)^m$ is the highest power of $x - r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:

$$\frac{A_1}{(x-r)} + \frac{A_2}{(x-r)^2} + \cdots + \frac{A_m}{(x-r)^m}.$$

Do this for each distinct linear factor of $g(x)$.

2. Let $x^2 + px + q$ be an irreducible quadratic factor of $g(x)$ so that $x^2 + px + q$ has no real roots. Suppose that $(x^2+px+q)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1x + C_1}{(x^2+px+q)} + \frac{B_2x + C_2}{(x^2+px+q)^2} + \cdots + \frac{B_nx + C_n}{(x^2+px+q)^n}.$$

Do this for each distinct quadratic factor of $g(x)$.

3. Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.
4. Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

Example

Use partial fractions to evaluate

$$\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx.$$

Solution:

The partial fraction decomposition has the form

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3}.$$

$$\begin{aligned} &x^2+4x+1 \\ &= A(x+1)(x+3) + B(x-1)(x+3) + C(x-1)(x+1) \\ &= A(x^2+4x+3) + B(x^2+2x-3) + C(x^2-1) \\ &= (A+B+C)x^2 + (4A+2B)x + (3A-3B-C). \end{aligned}$$

$$\begin{aligned} &\text{Coefficient of } x^2: && A + B + C = 1 \\ &\text{Coefficient of } x^1: && 4A + 2B \quad\;\; = 4 \\ &\text{Coefficient of } x^0: && 3A - 3B - C = 1 \end{aligned}$$

The solution is

$$A = 3/4,\ B = 1/2, \text{ and } C = -1/4.$$

Hence we have

$$\begin{aligned} &\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx \\ &= \int \left[\frac{3}{4}\frac{1}{x-1} + \frac{1}{2}\frac{1}{x+1} - \frac{1}{4}\frac{1}{x+3}\right] dx \\ &= \frac{3}{4}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x+3| + K \end{aligned}$$

Example

Use partial fractions to evaluate

$$\int \frac{6x+7}{(x+2)^2}\,dx.$$

Solution:

$$\frac{6x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$$

$$\begin{aligned} 6x + 7 &= A(x+2) + B \\ &= Ax + (2A + B) \end{aligned}$$

Equating coefficients of corresponding powers of *x* gives

$$A = 6 \qquad \text{and} \qquad B = -5.$$

Therefore,

$$\begin{aligned} \int \frac{6x+7}{(x+2)^2}\,dx &= \int \left(\frac{6}{x+2} - \frac{5}{(x+2)^2}\right) dx \\ &= 6\int \frac{dx}{x+2} - 5\int (x+2)^{-2}\,dx \\ &= 6\ln|x+2| + 5(x+2)^{-1} + C. \end{aligned}$$

Example

Use partial fractions to evaluate

$$\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}\,dx.$$

Solution:

$$\begin{array}{r@{\,}l} & \quad 2x \\ x^2-2x-3 \,\big)\!\!\!\! & \overline{\,2x^3 - 4x^2 - x - 3} \\ & \underline{\,2x^3 - 4x^2 - 6x\phantom{{}-3}} \\ & \qquad\qquad\;\; 5x - 3 \end{array}$$

Write the improper fraction as a polynomial plus a proper fraction.

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = 2x + \frac{5x - 3}{x^2 - 2x - 3}$$

$$\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\,dx$$

$$= \int 2x\,dx + \int \frac{5x - 3}{x^2 - 2x - 3}\,dx$$

$$= \int 2x\,dx + \int \frac{2}{x + 1}\,dx + \int \frac{3}{x - 3}\,dx$$

$$= x^2 + 2\ln|x + 1| + 3\ln|x - 3| + C.$$

Example

Use partial fractions to evaluate

$$\int \frac{-2x + 4}{(x^2 + 1)(x - 1)^2}\,dx.$$

Solution:
The denominator has an irreducible quadratic factor as well as a repeated linear factor, so we write

$$\frac{-2x + 4}{(x^2 + 1)(x - 1)^2} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x - 1} + \frac{D}{(x - 1)^2}.$$

Clearing the equation of fractions gives

$$\begin{aligned}-2x + 4 &= (Ax + B)(x - 1)^2 + C(x - 1)(x^2 + 1) + D(x^2 + 1)\\ &= (A + C)x^3 + (-2A + B - C + D)x^2\\ &\quad + (A - 2B + C)x + (B - C + D).\end{aligned}$$

Equating coefficients of like terms gives

Coefficients of x^3: $0 = A + C$

Coefficients of x^2: $0 = -2A + B - C + D$

Coefficients of x^1: $-2 = A - 2B + C$

Coefficients of x^0: $4 = B - C + D$

Solve these equations simultaneously to find the values of *A*, *B*, *C*, and *D*:

$$\begin{aligned}-4 &= -2A, \qquad A = 2\\ C &= -A = -2\\ B &= (A + C + 2)/2 = 1\\ D &= 4 - B + C = 1.\end{aligned}$$

Obtaining

$$\frac{-2x + 4}{(x^2 + 1)(x - 1)^2} = \frac{2x + 1}{x^2 + 1} - \frac{2}{x - 1} + \frac{1}{(x - 1)^2}.$$

Finally, using the expansion above we can integrate:

$$\int \frac{-2x + 4}{(x^2 + 1)(x - 1)^2}\,dx$$

$$= \int \left(\frac{2x + 1}{x^2 + 1} - \frac{2}{x - 1} + \frac{1}{(x - 1)^2}\right)dx$$

$$= \int \left(\frac{2x}{x^2 + 1} + \frac{1}{x^2 + 1} - \frac{2}{x - 1} + \frac{1}{(x - 1)^2}\right)dx$$

$$= \ln(x^2 + 1) + \tan^{-1} x - 2\ln|x - 1| - \frac{1}{x - 1} + C.$$

In some problems, assigning small values to *x*, such as *x* = 0, ±1, ±2, to get equations in *A*, *B*, and *C* provides a fast alternative to other methods.

Example

Find *A*, *B*, and *C* in the expression

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

by assigning numerical values to *x*.

Solution Clear fractions to get

$$x^2 + 1 =$$

$$A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2).$$

Then let *x* = 1, 2, 3 successively to find *A*, *B*, and *C*:

$$\begin{aligned}
x = 1: \quad (1)^2 + 1 &= A(-1)(-2) + B(0) + C(0)\\
2 &= 2A\\
A &= 1\\
x = 2: \quad (2)^2 + 1 &= A(0) + B(1)(-1) + C(0)\\
5 &= -B\\
B &= -5\\
x = 3: \quad (3)^2 + 1 &= A(0) + B(0) + C(2)(1)\\
10 &= 2C\\
C &= 5.
\end{aligned}$$

Conclusion:

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{1}{x-1} - \frac{5}{x-2} + \frac{5}{x-3}.$$

Exercises

1 – 8: Expand the quotients by partial fractions.

1. $\dfrac{5x-13}{(x-3)(x-2)}$
2. $\dfrac{5x-7}{x^2-3x+2}$
3. $\dfrac{x+4}{(x+1)^2}$
4. $\dfrac{2x+2}{x^2-2x+1}$
5. $\dfrac{z+1}{z^2(z-1)}$
6. $\dfrac{z}{z^3-z^2-6z}$
7. $\dfrac{t^2+8}{t^2-5t+6}$
8. $\dfrac{t^4+9}{t^4+9t^2}$

---

9 – 12: Express the integrand as a sum of partial fractions and evaluate the integrals.

9. $\displaystyle\int \frac{dx}{1-x^2}$

10. $\displaystyle\int \frac{dx}{x^2+2x}$

11. $\displaystyle\int \frac{x+4}{x^2+5x-6}\,dx$

12. $\displaystyle\int \frac{2x+1}{x^2-7x+12}\,dx$

13 – 14: Express the integrand as a sum of partial fractions and evaluate the integrals.

13. $\displaystyle\int \frac{dx}{(x^2-1)^2}$

14. $\displaystyle\int \frac{x^2\,dx}{(x-1)(x^2+2x+1)}$

---

15 – 18: Express the integrand as a sum of partial fractions and evaluate the integrals.

15. $\displaystyle\int \frac{y^2+2y+1}{(y^2+1)^2}\,dy$

16. $\displaystyle\int \frac{8x^2+8x+2}{(4x^2+1)^2}\,dx$

17. $\displaystyle\int \frac{2s+2}{(s^2+1)(s-1)^3}\,ds$

18. $\displaystyle\int \frac{s^4+81}{s(s^2+9)^2}\,ds$

Meeting 12

Integral - 4

Integration by Parts

Integration by parts is a technique for simplifying integrals of the form

$$\int f(x)g(x)\,dx.$$

Product Rule in Integral Form

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

$$\int \frac{d}{dx}[f(x)g(x)]\,dx = \int [f'(x)g(x) + f(x)g'(x)]\,dx$$

$$\int \frac{d}{dx}[f(x)g(x)]\,dx = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

$$\int f(x)g'(x)\,dx = \int \frac{d}{dx}[f(x)g(x)]\,dx - \int f'(x)g(x)\,dx$$

$$\boxed{\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx}$$

Integration by Parts Formula

$$\boxed{\int u\,dv = uv - \int v\,du}$$

With a proper choice of *u* and *v*, the second integral may be easier to evaluate than the first. In using the formula, various choices may be available for *u* and *dv*. To avoid mistakes, we always list our choices for *u* and *dv*, then we add to the list our calculated new terms *du* and *v*, and finally we apply the formula of integration by parts.

Example

Find

$$\int x\cos x\,dx.$$

Solution We choose and get

$$\begin{aligned} u &= x, & dv &= \cos x\,dx, \\ du &= dx, & v &= \sin x. \end{aligned}$$

Then,

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx$$
$$= x\sin x + \cos x + C.$$

The goal of integration by parts is to go from an integral $\int u\,dv$ that we don't see how to evaluate to an integral $\int v\,du$ that we can evaluate.

When finding *v* from *dv*, any antiderivative will work and we usually pick the simplest one

Example

Find

$$\int \ln x\,dx.$$

Solution We choose and get

$$u = \ln x \qquad dv = dx$$
$$du = \frac{1}{x}\,dx, \qquad v = x.$$

Then,

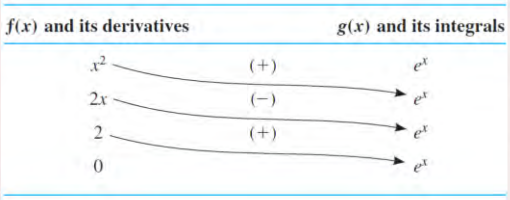
$$\int \ln x\,dx = x\ln x - \int x\cdot\frac{1}{x}\,dx$$
$$= x\ln x - \int dx$$
$$= x\ln x - x + C.$$

Tabular Integration Can Simplify Repeated Integrations

Find

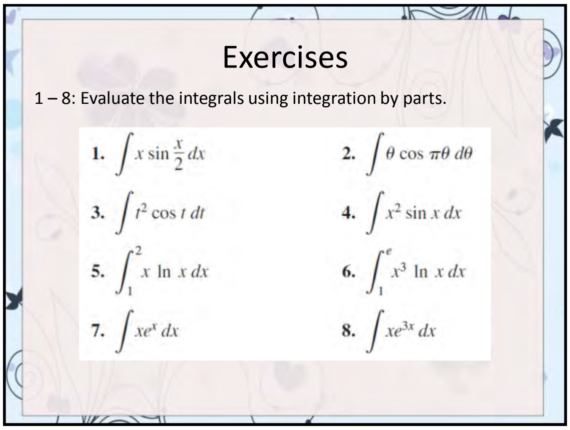
$$\int x^2e^x\,dx.$$

Solution



$f(x)$ and its derivatives		$g(x)$ and its integrals
x^2	$(+)$	e^x
$2x$	$(-)$	e^x
2	$(+)$	e^x
0		e^x

$$\int x^2e^x\,dx = x^2e^x - 2xe^x + 2e^x + C.$$



Exercises

1 – 8: Evaluate the integrals using integration by parts.

1. $\int x \sin \frac{x}{2}\, dx$

2. $\int \theta \cos \pi\theta\, d\theta$

3. $\int t^2 \cos t\, dt$

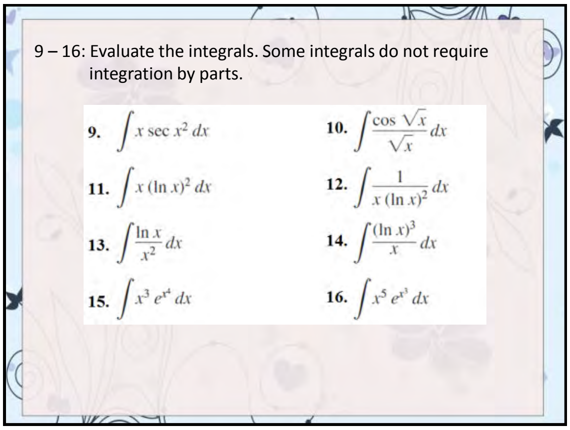
4. $\int x^2 \sin x\, dx$

5. $\int_1^2 x \ln x\, dx$

6. $\int_1^e x^3 \ln x\, dx$

7. $\int xe^x\, dx$

8. $\int xe^{3x}\, dx$



9 – 16: Evaluate the integrals. Some integrals do not require integration by parts.

9. $\int x \sec x^2\, dx$

10. $\int \frac{\cos \sqrt{x}}{\sqrt{x}}\, dx$

11. $\int x (\ln x)^2\, dx$

12. $\int \frac{1}{x (\ln x)^2}\, dx$

13. $\int \frac{\ln x}{x^2}\, dx$

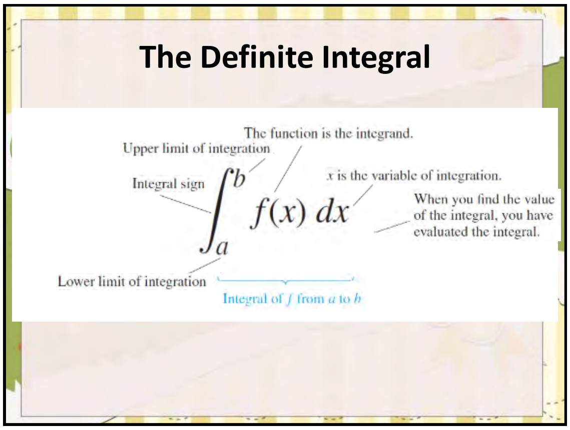
14. $\int \frac{(\ln x)^3}{x}\, dx$

15. $\int x^3 e^{x^4}\, dx$

16. $\int x^5 e^{x^3}\, dx$

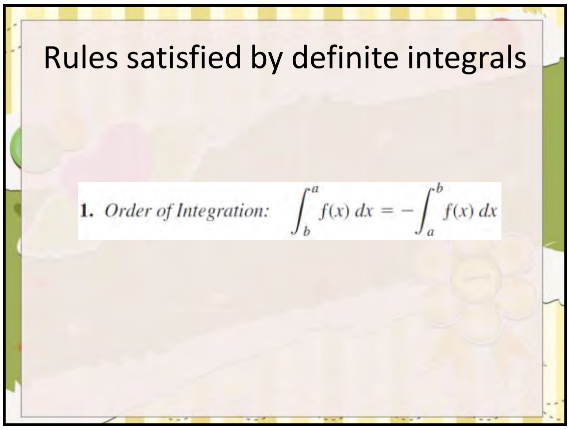
Meeting 13

Integral - 5



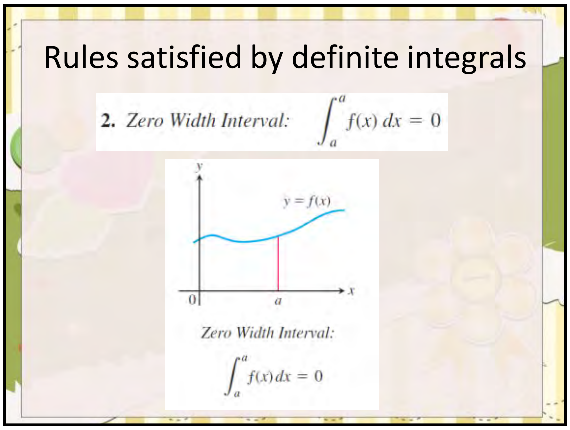
The Definite Integral

$$\int_a^b f(x)\,dx$$



Rules satisfied by definite integrals

1. *Order of Integration:* $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

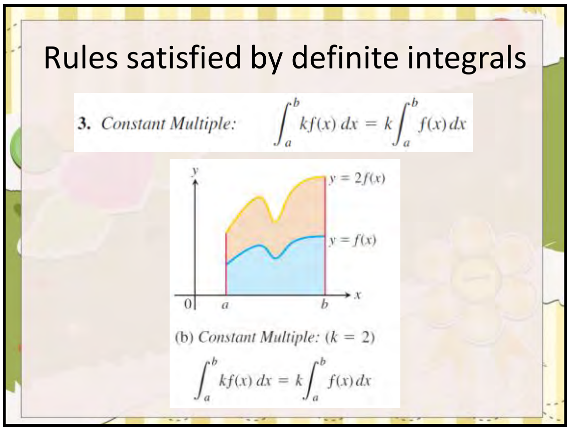


Rules satisfied by definite integrals

2. *Zero Width Interval:* $\int_a^a f(x)\,dx = 0$

Zero Width Interval:

$$\int_a^a f(x)\,dx = 0$$

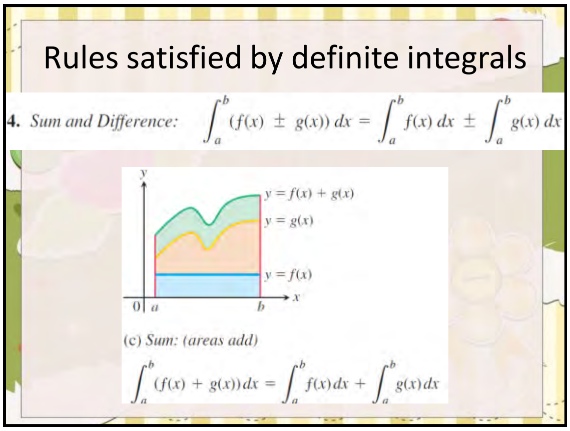


Rules satisfied by definite integrals

3. *Constant Multiple:* $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

(b) *Constant Multiple:* $(k = 2)$

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$

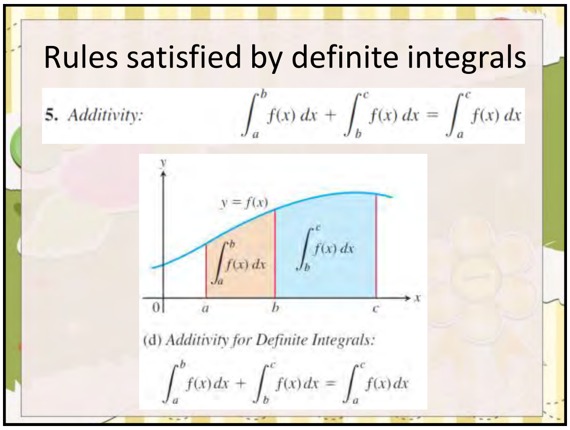


Rules satisfied by definite integrals

4. *Sum and Difference:* $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

(c) *Sum: (areas add)*

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

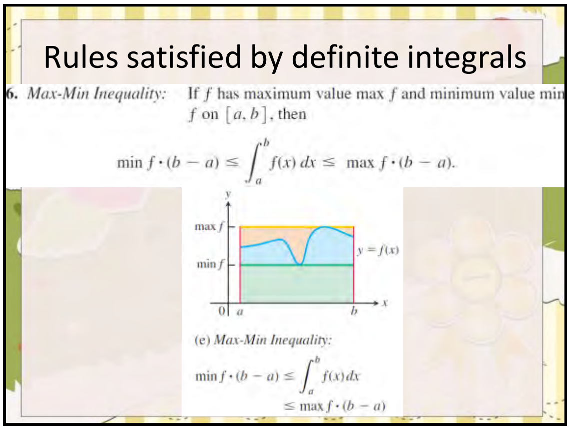


Rules satisfied by definite integrals

5. *Additivity:* $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

(d) *Additivity for Definite Integrals:*

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$



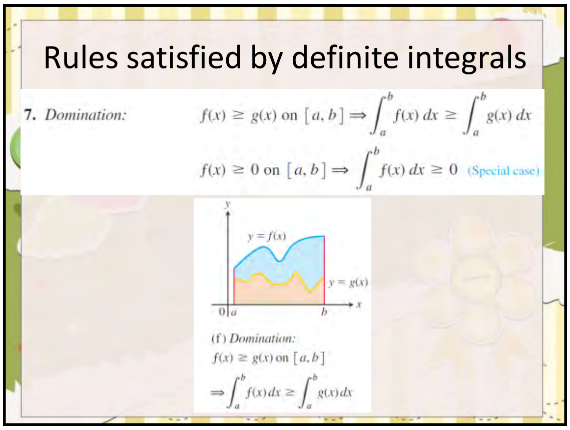
Rules satisfied by definite integrals

6. *Max-Min Inequality:* If f has maximum value max f and minimum value min f on $[a, b]$, then

$$\min f \cdot (b - a) \le \int_a^b f(x)\,dx \le \max f \cdot (b - a).$$

(e) *Max-Min Inequality:*

$$\min f \cdot (b - a) \le \int_a^b f(x)\,dx$$
$$\le \max f \cdot (b - a)$$

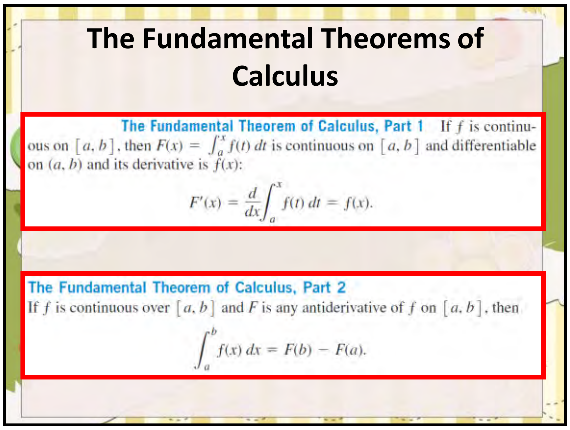


Rules satisfied by definite integrals

7. *Domination:* $f(x) \ge g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

$f(x) \ge 0$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge 0$ (Special case)

(f) *Domination:*
$f(x) \ge g(x)$ on $[a, b]$
$\Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$



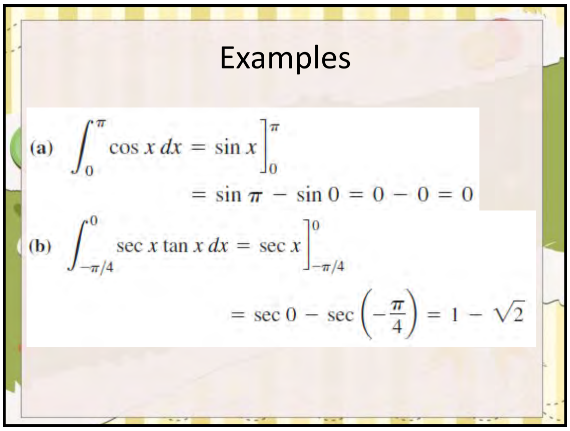
The Fundamental Theorems of Calculus

The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$:

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

The Fundamental Theorem of Calculus, Part 2
If f is continuous over $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

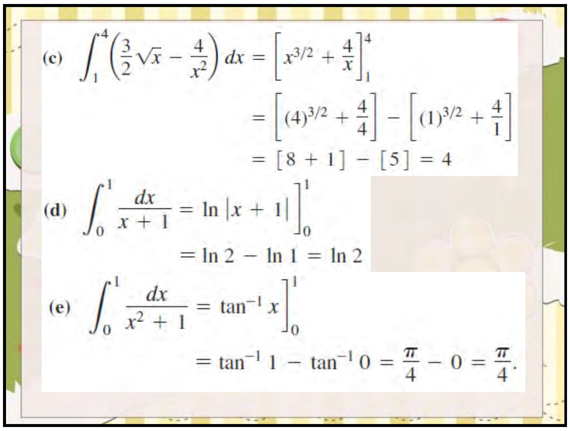
$$\int_a^b f(x)\,dx = F(b) - F(a).$$



Examples

(a) $$\int_0^\pi \cos x\,dx = \sin x\Big]_0^\pi$$
$$= \sin \pi - \sin 0 = 0 - 0 = 0$$

(b) $$\int_{-\pi/4}^0 \sec x \tan x\,dx = \sec x\Big]_{-\pi/4}^0$$
$$= \sec 0 - \sec\left(-\frac{\pi}{4}\right) = 1 - \sqrt{2}$$



(c) $$\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx = \left[x^{3/2} + \frac{4}{x}\right]_1^4$$
$$= \left[(4)^{3/2} + \frac{4}{4}\right] - \left[(1)^{3/2} + \frac{4}{1}\right]$$
$$= [8 + 1] - [5] = 4$$

(d) $$\int_0^1 \frac{dx}{x + 1} = \ln |x + 1|\Big]_0^1$$
$$= \ln 2 - \ln 1 = \ln 2$$

(e) $$\int_0^1 \frac{dx}{x^2 + 1} = \tan^{-1} x\Big]_0^1$$
$$= \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}.$$

Exercises

1 – 10: Evaluate the integrals.

1. $\int_0^2 x(x-3)\,dx$
2. $\int_{-1}^1 (x^2-2x+3)\,dx$
3. $\int_{-2}^2 \frac{3}{(x+3)^4}\,dx$
4. $\int_{-1}^1 x^{299}\,dx$
5. $\int_1^4 \left(3x^2-\frac{x^3}{4}\right)dx$
6. $\int_{-2}^3 (x^3-2x+3)\,dx$
7. $\int_0^1 (x^2+\sqrt{x})\,dx$
8. $\int_1^{32} x^{-6/5}\,dx$
9. $\int_0^{\pi/3} 2\sec^2 x\,dx$
10. $\int_0^{\pi} (1+\cos x)\,dx$

Substitution in Definite Integrals

Substitution in Definite Integrals If g' is continuous on the interval $[a, b]$ and f is continuous on the range of $g(x) = u$, then

$$\int_a^b f(g(x))\cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Example

Evaluate

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx.$$

Solution

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx = \int_0^2 \sqrt{u}\,du$$

Let $u = x^3+1$, $du = 3x^2\,dx$.
When $x=-1$, $u=(-1)^3+1=0$.
When $x=1$, $u=(1)^3+1=2$.

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx = \int_0^2 \sqrt{u}\,du$$

$$= \frac{2}{3}u^{3/2}\Big]_0^2$$

$$= \frac{2}{3}\left[2^{3/2}-0^{3/2}\right]$$

$$= \frac{2}{3}\left[2\sqrt{2}\right]$$

$$= \frac{4\sqrt{2}}{3}$$

Alternative method

Transform the integral as an indefinite integral, integrate, change back to *x*, and use the original *x*-limits.

$$\int 3x^2\sqrt{x^3+1}\,dx = \int \sqrt{u}\,du$$

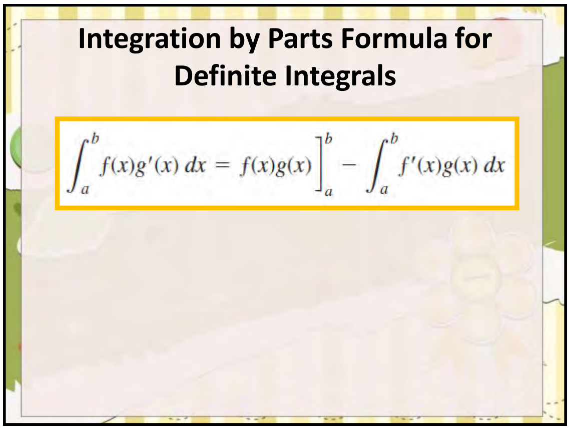
$$= \frac{2}{3}u^{3/2}+C$$

$$= \frac{2}{3}(x^3+1)^{3/2}+C$$

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx = \frac{2}{3}(x^3+1)^{3/2}\Big]_{-1}^{1}$$

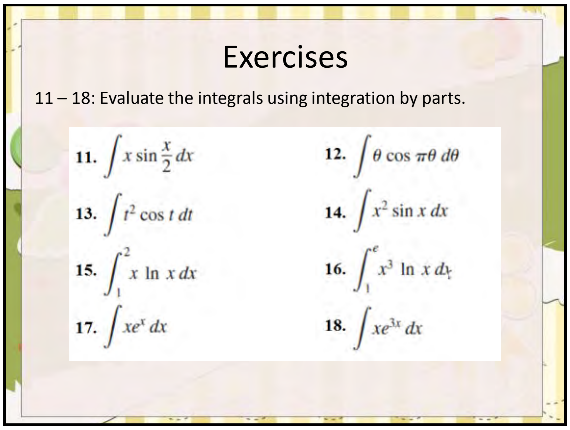
$$= \frac{2}{3}\left[((1)^3+1)^{3/2}-((-1)^3+1)^{3/2}\right]$$

$$= \frac{2}{3}\left[2^{3/2}-0^{3/2}\right] = \frac{2}{3}\left[2\sqrt{2}\right] = \frac{4\sqrt{2}}{3}$$



Integration by Parts Formula for Definite Integrals

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)\,dx$$



Exercises

11 – 18: Evaluate the integrals using integration by parts.

11. $\int x \sin \frac{x}{2}\,dx$

12. $\int \theta \cos \pi\theta\,d\theta$

13. $\int t^2 \cos t\,dt$

14. $\int x^2 \sin x\,dx$

15. $\int_1^2 x \ln x\,dx$

16. $\int_1^e x^3 \ln x\,dx$

17. $\int xe^x\,dx$

18. $\int xe^{3x}\,dx$

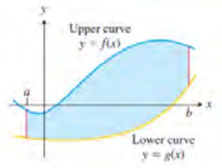
Meeting 14

Application of Integral

Area of the Region between the Curves

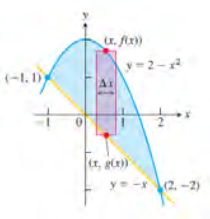
DEFINITION If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b is the integral of $(f - g)$ from a to b:

$$A = \int_a^b [f(x) - g(x)]\, dx.$$



Example

Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.



Solution

$$\begin{aligned} 2 - x^2 &= -x \\ x^2 - x - 2 &= 0 \\ (x+1)(x-2) &= 0 \\ x = -1, \quad & x = 2. \end{aligned}$$

The area between the curves is

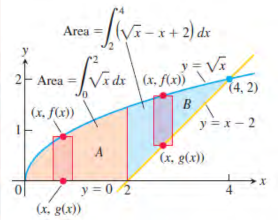
$$A = \int_a^b [f(x) - g(x)]\, dx = \int_{-1}^{2} [(2 - x^2) - (-x)]\, dx$$

$$= \int_{-1}^{2} (2 + x - x^2)\, dx = \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{2}$$

$$= \left(4 + \frac{4}{2} - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} + \frac{1}{3}\right) = \frac{9}{2}.$$

Example

Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.



Solution

For $0 \leq x \leq 2$: $\quad f(x) - g(x) = \sqrt{x} - 0 = \sqrt{x}$

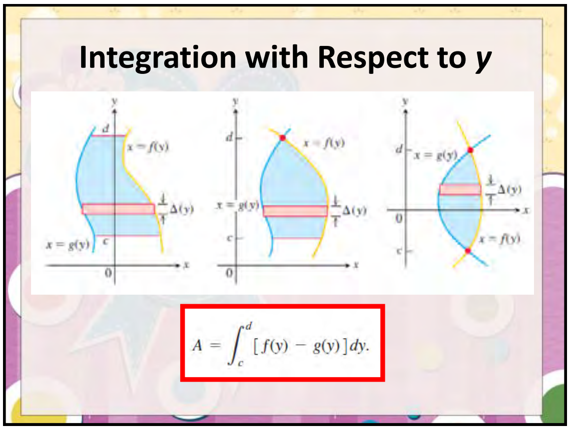
For $2 \leq x \leq 4$: $\quad f(x) - g(x) = \sqrt{x} - (x - 2) = \sqrt{x} - x + 2$

$$\text{Total area} = \underbrace{\int_0^2 \sqrt{x}\, dx}_{\text{area of } A} + \underbrace{\int_2^4 (\sqrt{x} - x + 2)\, dx}_{\text{area of } B}$$

$$= \left[\frac{2}{3}x^{3/2}\right]_0^2 + \left[\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\right]_2^4$$

$$= \frac{2}{3}(2)^{3/2} - 0 + \left(\frac{2}{3}(4)^{3/2} - 8 + 8\right) - \left(\frac{2}{3}(2)^{3/2} - 2 + 4\right)$$

$$= \frac{2}{3}(8) - 2 = \frac{10}{3}.$$



Integration with Respect to *y*

$$A = \int_c^d [f(y) - g(y)]\,dy.$$

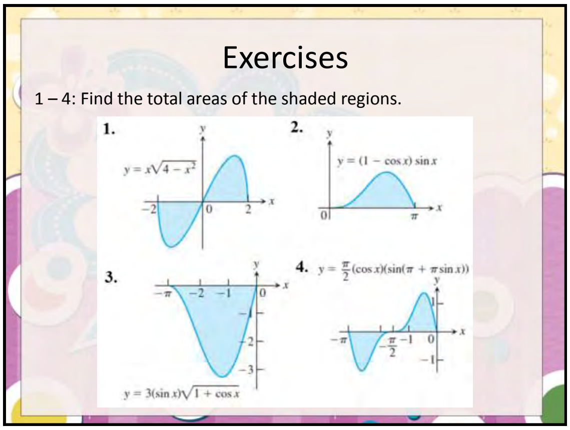
Example

Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the *x*-axis and the line *y* = *x* – 2 by integrating with respect to *y*.

Solution

$$y + 2 = y^2$$
$$y^2 - y - 2 = 0$$
$$(y + 1)(y - 2) = 0$$
$$y = -1, \quad y = 2$$

$$A = \int_c^d [f(y) - g(y)]\,dy = \int_0^2 [y + 2 - y^2]\,dy$$
$$= \int_0^2 [2 + y - y^2]\,dy$$
$$= \left[2y + \frac{y^2}{2} - \frac{y^3}{3}\right]_0^2$$
$$= 4 + \frac{4}{2} - \frac{8}{3} = \frac{10}{3}.$$



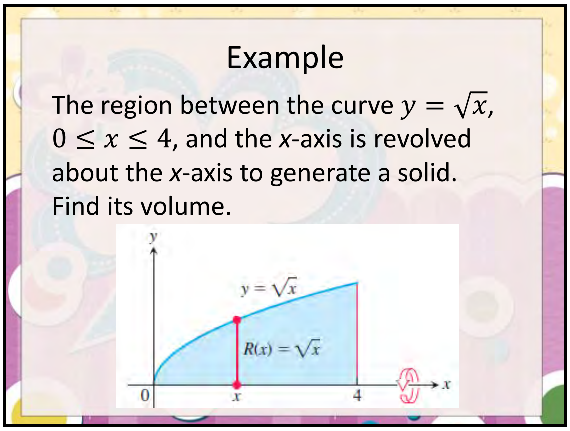
Exercises

1 – 4: Find the total areas of the shaded regions.

Volume by Disks for Rotation About the *x*-axis

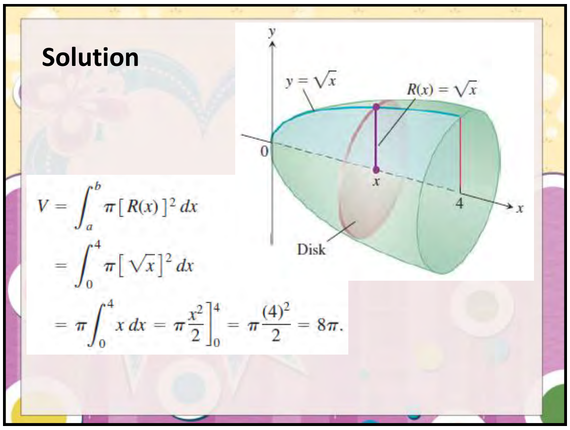
The solid generated by rotating (or revolving) a plane region about an axis in its plane is called a **solid of revolution**.

$$V = \int_a^b A(x)\,dx = \int_a^b \pi\,[R(x)]^2\,dx.$$



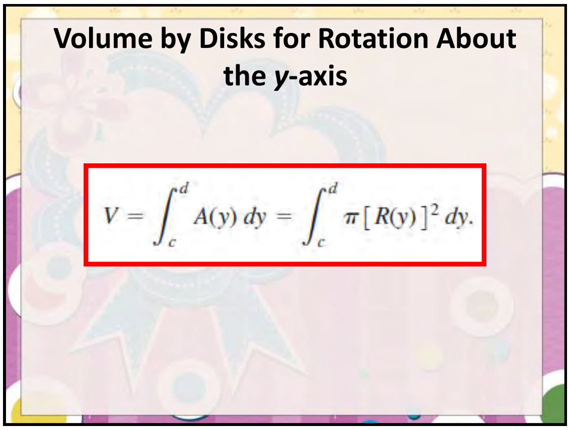
Example

The region between the curve $y = \sqrt{x}$, $0 \leq x \leq 4$, and the *x*-axis is revolved about the *x*-axis to generate a solid. Find its volume.



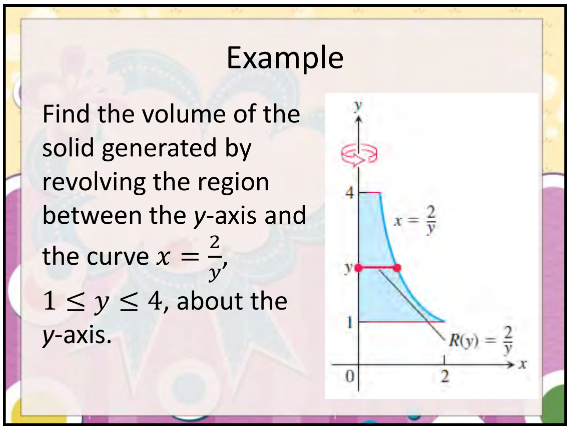
Solution

$$V=\int_a^b \pi[R(x)]^2\,dx$$
$$=\int_0^4 \pi[\sqrt{x}]^2\,dx$$
$$=\pi\int_0^4 x\,dx=\pi\frac{x^2}{2}\Big]_0^4=\pi\frac{(4)^2}{2}=8\pi.$$



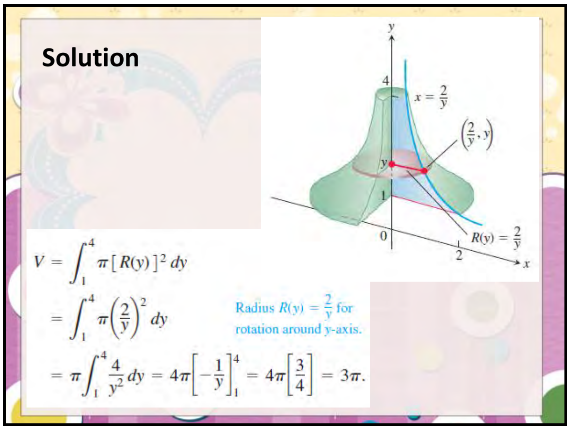
Volume by Disks for Rotation About the *y*-axis

$$\boxed{V=\int_c^d A(y)\,dy=\int_c^d \pi[R(y)]^2\,dy.}$$



Example

Find the volume of the solid generated by revolving the region between the y-axis and the curve $x=\dfrac{2}{y}$, $1\le y\le 4$, about the y-axis.

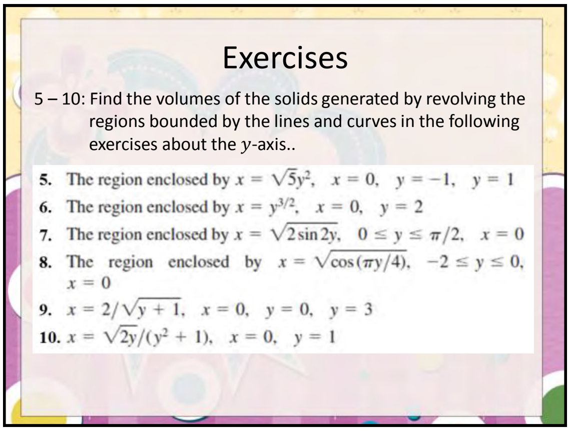


Solution

$$V=\int_1^4 \pi[R(y)]^2\,dy$$
$$=\int_1^4 \pi\left(\frac{2}{y}\right)^2 dy$$

Radius $R(y)=\frac{2}{y}$ for rotation around y-axis.

$$=\pi\int_1^4 \frac{4}{y^2}\,dy=4\pi\left[-\frac{1}{y}\right]_1^4=4\pi\left[\frac{3}{4}\right]=3\pi.$$



Exercises

5 – 10: Find the volumes of the solids generated by revolving the regions bounded by the lines and curves in the following exercises about the y-axis..

5. The region enclosed by $x=\sqrt{5}y^2$, $x=0$, $y=-1$, $y=1$

6. The region enclosed by $x=y^{3/2}$, $x=0$, $y=2$

7. The region enclosed by $x=\sqrt{2\sin 2y}$, $0\le y\le \pi/2$, $x=0$

8. The region enclosed by $x=\sqrt{\cos(\pi y/4)}$, $-2\le y\le 0$, $x=0$

9. $x=2/\sqrt{y+1}$, $x=0$, $y=0$, $y=3$

10. $x=\sqrt{2y}/(y^2+1)$, $x=0$, $y=1$